Stabilisation des systèmes quantiques à temps discrets et stabilité des filtres quantiques à temps continus

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Introduction

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La théorie du contrôle quantique s'est largement développée au cours de ces trois dernières décennies. Ce champ de recherche concerne le contrôle du comportement de systèmes physiques qui obéissent aux lois de la mécanique quantique. La capacité à contrôler des systèmes quantiques est en passe de devenir une étape essentielle dans l'émergence de technologies liées à l'informatique quantique, la cryptographie quantique, et la métrologie de haute précision. Comme ce domaine émergent comporte un grand nombre d'applications en chimie quantique [81, 31, 105], information quantique [76, 106], métrologie [94, 38], résonance magnétique nucléaire (RMN) [54, 89] et optique quantique [110, 104], il rassemble autour de sujets communs des physiciens, des chimistes, des mathématiciens et des ingénieurs.

Les phénomènes quantiques tels que la superposition ou l'intrication d'états ne se produisent pas dans la physique classique. Ces représentations non-classiques sont uniquement observables à des échelles de temps qui sont courtes en comparaison de celles qui caractérisent les interactions dans un environnement classique.

La manipulation de systèmes quantiques permet de réaliser des tâches qui sont hors de portée des appareils classiques. Ceci repose intrinsèquement sur la nature de l'information quantique : les ordinateurs quantiques dépasseraient significativement les machines classiques dans la résolution de certains problèmes [75]. Cependant, même si des progrès substantiels ont été réalisés récemment, de sérieuses difficultés restent à l'ordre du jour, parmi lesquelles la décohérence est certainement la principale. Les systèmes complexes constitués de nombreux qubits doivent être préparés dans des états quantiques fragiles qui sont rapidement détruits suite à leur couplage inévitable avec leur environnement. En effet, la décohérence est la perte d'information dans le système suite au couplage avec son environnement. Le système en interaction avec son environnement obéit à l'équation de Schrödinger. Si l'on trace sur l'environnement, la dynamique du système est irréversible, c'est-à-dire qu'elle ne peut pas être décrite par une transformation unitaire, cependant l'évolution du système total, c'est-à-dire, le système plus l'environnement, peut être décrit par une transformation unitaire.

La préservation du système contre la décohérence est une tâche majeure pour mener à bien les applications citées précédemment. Dans le domaine de l'information quantique, des codes de correction d'erreur quantique ont été mis au point pour compenser de telles pertes d'information au contact de l'environnement. Comme expliqué dans [75], de tels codes combinent des transformations unitaires, suivies par des mesures de qubits dont les issues conditionnent une transformation unitaire supplémentaire visant à compenser une éventuelle décohérence affectant un qubit. Ainsi, les codes de correction d'erreur quantiques reposent sur une boucle de rétroaction et sont liés à des questions portant sur la théorie du contrôle des systèmes et la stabilisation en boucle fermée [1]. Dans cette thèse, nous-nous pencherons sur la stabilisation en boucle fermée des systèmes quantiques, dans le but de compenser la décohérence.

Pour ce qui est de l'introduction mathématique au contrôle des systèmes quantiques, nous-nous référons au manuel récent [30]. Pour ce qui relève de la physique, nous renvoyons à un autre manuel récent [112]. Un cours d'introduction générale à la mécanique quantique (systèmes de spin demi-entier et oscillateur harmonique quantique) est référencé en [83] et une exposition plus avancée est donnée dans [44]. Les notions de contrôle non-linéaires sont introduites dans le manuel classique [53] dédié aux systèmes de dimension finie, et dans [28] qui traite des systèmes de dimension infinie régis par des équations aux dérivées partielles.

Quantum control theory has been widely developed through the last three decades. This field is concerned with controlling the behavior of physical systems which obey the laws of quantum mechanics. The ability to control quantum systems is becoming an essential step towards emerging technologies such as quantum computation, quantum cryptography and high precision metrology. As this emerging field has a wide range of applications in quantum chemistry [81, 31, 105], quantum information [76, 106], metrology [94, 38], nuclear magnetic resonance (NMR) [54, 89] and quantum optics [110, 104], it brings together physicists, chemists, mathematicians, and engineers.

Quantum phenomena like superposition and entanglement never happen in classical worlds. However, these non-classical phenomena are only observable in time scales that are short compared to those which characterize the coupling to the environment.

Manipulating quantum systems allows one to accomplish tasks far beyond the reach of classical devices. Quantum information is paradigmatic in this sense: quantum computers will substantially outperform classical machines for several problems [75]. Though significant progress has been made recently, severe difficulties still remain, among which decoherence is certainly the most important. Large systems consisting of many qubits must be prepared in fragile quantum states, which are rapidly destroyed by their unavoidable coupling to the environment. In fact, decoherence is the loss of information from the system coupled to its environment. The system of interest with its environment obey to Schrödinger equation. If we trace out the environment, the system's dynamic is irreversible, i.e., it cannot be described by a unitary transformation, meanwhile the evolution of the total system, i.e., the system plus the environment, can be described by a unitary transformation.

Fighting against the decoherence is a major task for all the mentioned applications. In the domain of quantum information, quantum error correction codes have been developed to compensate such loss of information into the environment. As explained in [75], such codes combine unitary transformations, followed by qubit measurements whose outcomes condition additional unitary transformation to compensate any qubit decoherence jump. Thus quantum error correction codes rely on a feedback loop and lead to problems in control system theory and feedback stabilization [1]. In this thesis, we will investigate feedback stabilization of quantum systems, in order to compensate decoherence.

For a mathematical introduction to the control of quantum systems, we refer to the recent textbook [30]. For a physical exposure, we refer to another recent textbook [112]. A general introduction course to quantum mechanics (spin-half system and quantum harmonic oscillator) could be found in [83] and more advanced exposure is given in [44]. For nonlinear control, we refer to the classical textbook [53] devoted to finite-dimensional systems and to [28] addressing infinite dimensional systems governed by partial differential equations.

1.1 Strategies for quantum control

We give now a brief introduction to some interesting issues from the control system point of view. It is based on two surveys: the first one [65] presents various applications such as NMR, trapped ions, cavity quantum electrodynamics (CQED); the second one is more recent [35] and with a control systems perspective. Below, we will focus on the three following strategies: open-loop control (coherent control), measurement-based feedback and coherent feedback (autonomous feedback).

1.1.1 Open-loop control

This type of control is related to motion planning, controllability, and optimal control issues. The main goal is to steer a system, via a time-varying control input, from a known initial state (usually a stable ground state) to a target state (usually an excited or entangled state). Such control is purely in open-loop, since it avoids all feedbacks and measurements (see e.g., [30]).

Here we give some important prototypes of such an open-loop strategy: the optimal control approach is applied in physical chemistry [98, 19, 57] and in NMR experiments [54, 66, 84]; Lyapunov-based control approach is applied for bilinear Schrödinger equation [70]; different controllability issues have been studied in [58, 13, 17, 12].

Below we give a brief description of learning control, a type of control which is designed as an open-loop strategy and is widely applied in the chemistry community.

Learning control is usually used in controlling the chemical reactions (see e.g., [80]). Each iteration of learning control contains three steps: first, from a known initial state, one applies an open-loop control input depending on tuning parameters; second, a population measurement is performed on the final state; third, a learning algorithm exploiting the actual and past population measurements updates tuning parameters of open-loop control for the next iteration. The learning algorithm is designed in order to achieve some control goals typically, for maximization of some specific population. The initial control input may be a well-designed or a random control input.

1.1.2 Measurement-based feedback

Usually, closed-loop control involves two steps. The first step consists of extracting information on the system state from the past control input values and measured output values. This first step can be seen as the estimation, or filtering, step. For quantum systems, it is known as quantum filtering, which corresponds to the best estimate of the system state conditioned on the past measurement outcomes and control inputs [102, 22]. The second step is devoted to the computation of the next control input value based on the current filter state obtained in the first step. There are two strategies to design such feedback law: stochastic optimal control as in [15, 23, 47]; Lyapunov-based control as in [71].

This type of feedback where the controller is classical, has been previously considered for preparing squeezed states [95], for preserving entanglement [113], and for quantum state reduction [103, 71].

Measurement-based feedback is difficult to implement in a digital way. The necessary computations of the quantum filter and feedback law must be realized in real-time which could be difficult, since the time scales are usually very short. Historically, such kinds of controllers involving filtering and feedback are implemented in an analogue way via classical electronic circuits [67].

Another difficulty of this type of feedback is purely attached to the quantum nature: any measurement necessarily modifies the system state. This is due to the back-action of the measurement process.

In fact, one measures an observable of the system, which is a mathematical presentation of a physical quantity (like momentum, position). Moreover, this observable is presented by a Hermitian operator. In quantum mechanics literature [44], the following traditional types of measurements are considered: projective measurements, positive operator valued measure (POVM), and quantum non-demolition (QND) measurements. Let us now describe very briefly these types of measurements. Projective measurement respects the two following properties: first, the result of the measurement is necessarily one of the eigenvalues of the observable; and second, the system is projected onto an eigenstate corresponding to the observed eigenvalue. In practice, we do not measure directly the system of interest, rather we consider a large system, which is composed of the system interacting with its environment. Formally, we consider a large Hilbert space consisting of the system state and the measurement apparatus (called meter) state. In such a way, the experimenter observes the effect of the system on the environment by projective measurements of an observable on the meter. Such measurement leads to a POVM. Now consider an observable for the system, which is measured through its coupling with a meter and a projective measurement of an observable on the meter. If an arbitrary eigenstate of such a physical observable for the system has not been affected by measurements, one has a QND measurement. A sufficient but not necessary condition for this is that the system observable and the total Hamiltonian of the system with its meter commute. [1]

In this thesis, we apply measurement-based feedbacks for stabilizing discrete-time quantum systems subject to QND measurements. In fact, feedback stabilization of quantum states is closely related to the concept of QND measurement. This is because one needs to make sure that the measurement itself is not changing the desired target state.

1.1.3 Coherent feedback

A quantum controller is connected to the original quantum system to form a closed-loop system, which is a composite system made up of the original system and the controller [60]. In other words, for coherent feedback: the sensor, controller, and actuator are quantum objects that interact coherently with the original system to be controlled. The fact that no real-time processing of the measurement output is needed can play an important role to improve feedback performance.

In [73], a coherent feedback has been tested experimentally to transfer some previously established correlations between two spins to an auxiliary spin. A coherent feedback scheme based on the H-infinity control of linear quantum stochastic systems is proposed in [49]. It was experimentally tested in [61]. In [41, 48], a theory of quantum circuit and network is addressed. The physical origin of this theory can be found in [37, Chapter 12].

Squeezing enhancement via coherent feedback has been studied in [42, 111]. This type of feedback has recently led to new proposals for quantum memories, quantum error correction, and ultra-low power classical photonic signal processing in the following series

[1] See Chapter 2 for more details about these types of measurements.

of papers [51, 52, 64, 62, 63].

1.2 Contributions

In this section, we describe the contributions of this thesis. In Part I, we consider measurement-based feedback to stabilize particular states of light called photon-number states (Fock states) in a cavity quantum electrodynamics (CQED) setup developed at Ecole Normale Supérieure (ENS) de Paris. This system is a discrete-time system. Quantum mechanics laws show that such a system is governed by a non-linear controlled Markov chain where the state corresponds to the density matrix [2] of the cavity field. The feedback design relies on stochastic Lyapunov techniques. Also, we study a stabilizing feedback applied to more generic discrete-time quantum systems which are subject to QND measurements. This part is based on the following publications [88, 5, 3] and preprint [6].

Part II is devoted to stability of continuous-time quantum filters. First, we prove the stability of a continuous-time filter which is driven only by a Wiener process. This work is based on the following publication [2]. Second, the dynamics of discrete-time optimal filter taking into account classical measurement errors, introduced in Part I, will be used to design heuristically the general continuous-time optimal filter. In the following work in progress [4], we aim to make the heuristic rules, applied to obtain the general continuous-time optimal filter, more rigorous. Furthermore, we also wish to prove the stability of such an optimal filter.

Now, we explain the contents of this section. Subsection 1.2.1 is devoted to a short presentation of the cavity quantum electrodynamics (CQED) and some recent works related to Fock states. In Subsection 1.2.2, we summarize the obtained results on the stabilization of Fock states. Subsection 1.2.3 presents the construction of new families of Lyapunov functions first for Fock states and then for generic discrete-time quantum systems, which are subject to QND measurements. Finally Subsection 1.2.4 summarizes our stability results of continuous-time quantum filter with classical measurement errors.

1.2.1 Cavity quantum electrodynamics (CQED)

Cavity quantum electrodynamics (CQED) is particularly suitable for handling and studying the interaction of atoms and photons (see e.g., [44, Chapter 5] for a description of such CQED systems). The light trapped in a cavity of high finesse, and the atoms interacting with it, form a system virtually isolated from the environment. Moreover, for the experimentalists, it is the strong coupling regime between photons and atoms that is of great interest. In this regime, the cavity and atom decoherence becomes negligible at the scale of the coupling between atoms and photons.

Under these conditions, particular states of the electromagnetic field could be prepared, such as the photon-number states (Fock states). Unlike a classical field, these states contain

[2] See the definition of density matrix (density operator) in Chapter 2.

a precise number of photons. Fock states are very fragile and it is difficult to protect them from the environment.

In 2006, J. M. Geremia [39] proposed an experimental setup which prepares photon number states using continuous non-destructive indirect measurements, and applying measurement-based feedback. In 2009, I. Dotsenko et al. [36] proposed a measurement-based feedback scheme for the preparation and protection of photon-number states of light trapped in a high-Q microwave cavity. This scheme has been considered to operate in the same CQED setup presented in [32]. The form of such feedback is similar to the one proposed in [71] but now, the QND measurements are monitored in a discrete-time manner. The fact that the measurements are discrete in time provides enough time to calculate a quantum filter. Such filter provides the complete information about the cavity state ρ conditioned on the observation history.

Figure 1.1 shows the experimental setup considered in [36]. Let us now describe rapidly this setup. The Rydberg atoms (which are considered as two-level atoms whose state is generated by $|g\rangle$ and $|e\rangle$ corresponding resp. to the ground and the excited state) come out of box B and cross the first low-Q Ramsey cavity R_1. This causes a unitary transformation of the states of the atoms. Then, these atoms come to enter in interaction with the cavity C. This passage causes also a unitary transformation of the total state of the composite system (cavity mode and atom). Here, it is assumed that the interaction between the cavity and the atom is dispersive which is necessary to achieve QND measurements. These atoms cross the low-Q Ramsey cavity R_2 which causes another unitary transformation of the atomic state. Finally, these Rydberg atoms will be detected in D. The control corresponds to a coherent displacement of complex amplitude α that is applied via the microwave source S between two atoms passages.

We note here that, unlike classical closed-loop control, the sensor is a quantum object, meanwhile, the controller and the actuator are classical objects.

1.2.2 Stabilization of Fock states in a microwave cavity (photon box)

In fact, the feedback proposed in [36] has been obtained after two following steps: the first step consists in providing a state estimation through quantum filtering [22, 23] of the observed outcomes of atomic measurements in the detector D. In the second step, the field state is corrected, by applying a coherent field pulse of amplitude α which is a function of the estimated state. This feedback law is based on Lyapunov techniques [71].

Let us now describe the mathematical model. The state to be stabilized is the cavity state. The underlying Hilbert space is $\mathcal{H} = \mathbb{C}^{n^{\max}+1}$ which is assumed to be finite-dimensional (truncations to $n^{\max}$ photons). It admits $(|0\rangle, |1\rangle, \ldots, |n^{\max}\rangle)$ as an orthonormal basis. Each basis vector $|n\rangle \in \mathbb{C}^{n^{\max}+1}$ corresponds to a pure state, called Fock state, with precisely n photons in the cavity, $n \in \{0, \ldots, n^{\max}\}$.

The state of the cavity C can be described by the density operator ρ which belongs to the following set of well-defined density matrices:

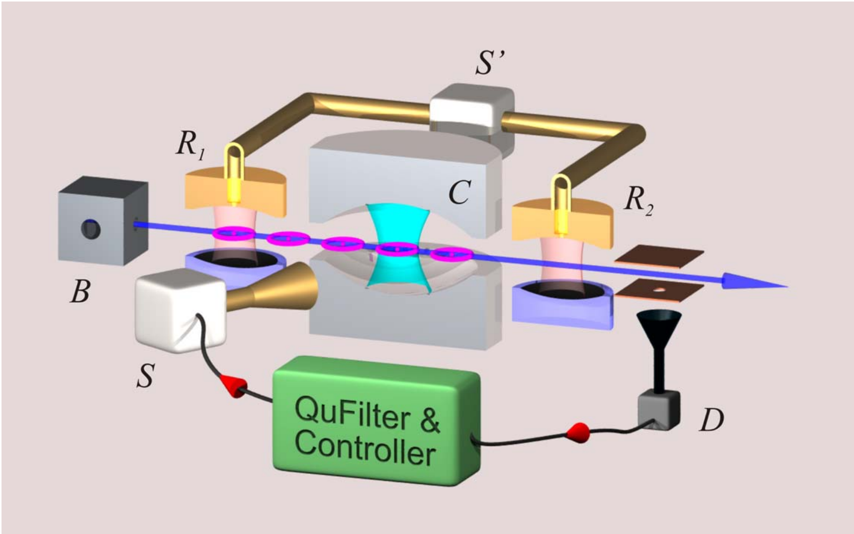

Figure 1.1: The ENS microwave cavity (photon box); atoms get out of box B one by one, undergo then a first Rabi pulse in Ramsey zone $R1$, become entangled with electromagnetic field trapped in C, undergo a second Rabi pulse in Ramsey zone $R2$ and finally are measured in the detector D. The control corresponds to a coherent displacement of amplitude $\alpha \in \mathbb{C}$ that is applied via the microwave source S between two atom passages [36].

$$\mathcal{X} = \left\{ \rho \in \mathbb{C}^{(n^{\max}+1)^2} \quad | \quad \rho = \rho^\dagger, \; \operatorname{Tr}(\rho) = 1, \; \rho \geq 0 \right\}. \tag{1.1}$$

If we take k as a time-index, the state of the cavity at step k is given by ρ_k. In an ideal experiment, injecting a control input α_k and a detection in $|e\rangle$ or $|g\rangle$ would actualize the state estimate into the following state (see [87, 44] for the model construction):

$$\rho_{k+1} = \mathbb{M}_{\mu_k}\big(\mathbb{D}_{\alpha_k}(\rho_k)\big). \tag{1.2}$$

Here

- the superoperator $\mathbb{M}_{\mu_k}$ is defined by $\mathbb{M}_{\mu_k}(\rho) := \frac{M_{\mu_k} \rho M_{\mu_k}^\dagger}{\operatorname{Tr}\left(M_{\mu_k} \rho M_{\mu_k}^\dagger\right)}$;

- μ_k denotes the random variable which corresponds to the outcomes of measurements which take values g with probability $\operatorname{Tr}\left(M_g(\mathbb{D}_{\alpha_k}(\rho_k))M_g^\dagger\right)$ and e with the complementary probability $\operatorname{Tr}\left(M_e(\mathbb{D}_{\alpha_k}(\rho_k))M_e^\dagger\right)$;

- the atoms-field interaction is dispersive, the measurement matrices take the following forms:
$$M_g = \cos\left(\varphi_0 \mathbb{1} + \vartheta \mathbf{N}\right) \quad \text{and} \quad M_e = \sin\left(\varphi_0 \mathbb{1} + \vartheta \mathbf{N}\right),$$
where $\mathbb{1}$ is the identity operator, $\mathbf{N} = \operatorname{diag}(0, 1, \ldots, n^{\max})$ is the photon number operator, and φ_0 and ϑ are constant parameters;

- the superoperator $\mathbb{D}_\alpha$ is defined by $\mathbb{D}_\alpha = D_\alpha \rho D_\alpha^\dagger$, where the control operator is given by the unitary transformation $D_\alpha = e^{\alpha \mathbf{a}^\dagger - \alpha^* \mathbf{a}}$. The annihilation operator is given by

$$\mathbf{a}\,|0\rangle = 0, \quad \mathbf{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle \text{ for } n = 1, \dots, n^{\max}.$$

Suppose that the measurement operators M_g and M_e satisfy the following assumption.

Assumption 1.1. *The constant parameters φ_0 and ϑ are chosen such that the measurement operators M_g and M_e are invertible and the spectrum of $M_g^\dagger M_g = M_g^2$ and $M_e^\dagger M_e = M_e^2$ are non degenerate.*

In open-loop case, when $\alpha = 0$, the control operator is given by $D_0 = \mathbb{1}$, where $\mathbb{1}$ denotes the identity operator. So ρ_{k+1} is described through the following dynamics

$$\rho_{k+1} = \frac{M_{\mu_k} \rho_k M_{\mu_k}^\dagger}{\mathrm{Tr}\left(M_{\mu_k} \rho_k M_{\mu_k}^\dagger\right)}. \tag{1.3}$$

Any Fock states $|m\rangle\langle m|$, with $|m\rangle = (\delta_{nm})_{n\in\{0,\dots,n^{\max}\}}$, is a fixed point of the open-loop dynamics. We prove the following theorem characterizing the open-loop behavior.

Theorem 1.1 (Theorem 3.1 (see page 57))**.** *Consider the Markov process ρ_k obeying* (1.3) *with an initial condition $\rho_0 \in \mathcal{X}$. Then*

- *for any $n \in \{0, \dots, n^{max}\}$, $\mathrm{Tr}\left(\rho_k |n\rangle\langle n|\right) = \langle n|\rho_k|n\rangle$ is a martingale;*
- *ρ_k converges with probability 1 to one of the $n^{max}+1$ Fock states $|n\rangle\langle n|$ with $n \in \{0, \dots, n^{max}\}$;*
- *the probability to converge towards the Fock state $|n\rangle\langle n|$ is given by $\mathrm{Tr}\left(\rho_0 |n\rangle\langle n|\right) = \langle n|\rho_0|n\rangle$.*

Indeed, the set $\{|m\rangle\langle m|\}_{m\in\{0,\dots,n^{\max}\}}$ is invariant under measurements when the control is zero. As a result, to ensure the stability towards the state $|\bar{n}\rangle\langle\bar{n}|$, for some $\bar{n} \in \{0, \dots, n^{\max}\}$, the natural Lyapunov function would be the following

$$V(\rho) = 1 - \mathrm{Tr}\left(\rho\,|\bar{n}\rangle\langle\bar{n}|\right),$$

since $\mathrm{Tr}\left(\rho\,|\bar{n}\rangle\langle\bar{n}|\right)$ defines an open-loop martingale. In [36, 68], in order to stabilize deterministically the Fock state $|\bar{n}\rangle\langle\bar{n}|$, a feedback law was found in such a way that the expectation value of the Lyapunov function $V(\rho)$ decreases in average, i.e., it defines a supermartingale.

The following theorem presents a feedback law which stabilizes the Markov chain (1.2) towards the state $|\bar{n}\rangle\langle\bar{n}|$.

Theorem 1.2 (Mirrahimi et al. [68]). *Consider* (1.2) *and suppose that Assumption 1.1 is satisfied. Take the following switching feedback*

$$\alpha_k = \begin{cases} \epsilon \mathrm{Tr}\left(\bar{\rho}\,[\rho_k, \mathbf{a}]\right) & \text{if } \mathrm{Tr}\left(\bar{\rho}\rho_k\right) \geq \varsigma \\ \underset{|\alpha|\leq\bar{\alpha}}{\mathit{argmax}}\left(\mathrm{Tr}\left(\bar{\rho}\ D_\alpha \rho_k D_\alpha^\dagger\right)\right) & \text{if } \mathrm{Tr}\left(\bar{\rho}\rho_k\right) < \varsigma \end{cases} \tag{1.4}$$

with ς, ϵ, $\bar{\alpha} > 0$ constants. Then for small enough ς and ϵ, the closed-loop trajectories converge almost surely towards the target Fock state $\bar{\rho} = |\bar{n}\rangle\langle\bar{n}|$. [3]

This feedback law was inspired from [71].

Taking delays into account

Theorem 1.2 ensures the convergence of the closed-loop system without any delay in the feedback loop. However, in the experimental setup [36], there exists a delay of τ steps between the measurement process and the feedback injection. Indeed, there are constantly, τ atoms flying between the cavity C to be controlled and the atom-detector D (typically $\tau = 5$). Therefore, in our feedback design, we do not have access to the measurement results for the τ last atoms.

The following dynamics describes the evolution of the cavity state taking into account a delay of τ steps (see [36]):

$$\rho_{k+1} = \mathbb{M}_{\mu_k}\left(\mathbb{D}_{\alpha_{k-\tau}}(\rho_k)\right). \tag{1.5}$$

In [3], we have considered this non-negligible delay in the design of a feedback law which stabilizes the above dynamics towards the state $|\bar{n}\rangle\langle\bar{n}|$. Indeed, we propose an adaptation of the quantum filter, based on a stochastic version of a Kalman-type predictor, which takes this delay into account by predicting the actual state of the system without having access to the result of the τ last detections. This compensation scheme was first introduced in [36].

In the following, we define the Kraus map $\mathbb{K}_\alpha(\rho)$.

Definition 1.1. *The Kraus map $\mathbb{K}_\alpha(\rho)$ is defined as follows,*

$$\mathbb{K}_\alpha(\rho) := M_g D_\alpha \rho D_\alpha^\dagger M_g^\dagger + M_e D_\alpha \rho D_\alpha^\dagger M_e^\dagger.$$

Moreover, we have $\mathbb{E}\left[\rho_{k+1}|\rho_k, \alpha_{k-\tau}\right] = \mathbb{K}_{\alpha_{k-\tau}}(\rho_k)$.

In order to stabilize the dynamics (1.5) towards the Fock state $|\bar{n}\rangle\langle\bar{n}|$, we cannot use the feedback law (1.4), since this is not causal. In fact, α_k depends on the state $\rho_{k+\tau}$. However, one can define the following state $(\rho_k, \alpha_{k-1}, \ldots, \alpha_{k-\tau})$ as the state at step k. This state defines a Markov chain. More precisely, denoting by $\chi = (\rho, \beta_1, \ldots, \beta_\tau)$, the state where β_l stands for the control α delayed by l steps. This Markov chain satisfies the following dynamics

[3] The notation $[A, B]$ refers to $AB - BA$.

$$\begin{cases} \rho_{k+1} &= \mathbb{M}_{\mu_k}(\mathbb{D}_{\beta_{\tau,k}}(\rho_k)) \\ \beta_{1,k+1} &= \alpha_k \\ \beta_{2,k+1} &= \beta_{1,k} \\ &\vdots \\ \beta_{\tau,k+1} &= \beta_{\tau-1,k}. \end{cases} \tag{1.6}$$

We prove the following theorem ensuring the convergence of the state χ_k towards the target state $\bar{\chi} = (\bar{\rho}, 0, \ldots, 0)$.

Theorem 1.3 (Theorem 3.2 (see page 64))**.** *Take the Markov chain* (1.6) *with the following feedback*

$$\alpha_k = \begin{cases} \epsilon \mathrm{Tr}\left(\bar{\rho}\ [\rho_k^{pred}, \mathbf{a}]\right) & \text{if } \mathrm{Tr}\left(\bar{\rho}\rho_k^{pred}\right) \geq \varsigma \\ \underset{|\alpha|\leq\bar{\alpha}}{\mathrm{argmax}} \left(\mathrm{Tr}\left(\bar{\rho}\ \mathbb{D}_\alpha(\rho_{g,k}^{pred})\right) \mathrm{Tr}\left(\bar{\rho}\ \mathbb{D}_\alpha(\rho_{e,k}^{pred})\right)\right) & \text{if } \mathrm{Tr}\left(\bar{\rho}\rho_k^{pred}\right) < \varsigma \end{cases} \tag{1.7}$$

with

$$\begin{cases} \rho_{g,k}^{pred} = \mathbb{K}_{\alpha_{k-1}} \circ \ldots \circ \mathbb{K}_{\alpha_{k-\tau+1}}(M_g D_{\alpha_{k-\tau}} \rho_k D_{\alpha_{k-\tau}}^\dagger M_g^\dagger) \\ \rho_{e,k}^{pred} = \mathbb{K}_{\alpha_{k-1}} \circ \ldots \circ \mathbb{K}_{\alpha_{k-\tau+1}}(M_e D_{\alpha_{k-\tau}} \rho_k D_{\alpha_{k-\tau}}^\dagger M_e^\dagger), \end{cases}$$

where $\rho_k^{pred} = \rho^{pred}(\chi_k) = \mathbb{E}\left[\rho_{k+\tau} \mid \chi_k\right] = \mathbb{K}_{\beta_{1,k}} \circ \ldots \circ \mathbb{K}_{\beta_{\tau,k}}(\rho_k)$ *and* $\bar{\alpha} > 0$*. Then, for small enough* $\epsilon > 0$ *and* $\varsigma > 0$*, the state* χ_k *converges almost surely towards* $\bar{\chi} = (\bar{\rho}, 0, \ldots, 0)$ *whatever the initial condition* $\chi_0 \in \mathcal{X} \times \mathbb{C}^\tau$ *is (the compact set* $\mathcal{X}$ *is defined by* (1.1)*).*

Moreover, in Proposition 3.2 (page 71), we study the convergence rate around the target state $|\bar{n}\rangle\langle\bar{n}|$. To this aim, we show that the largest Lyapunov exponent (see Definition A.3 in Appendix A) of the linearized closed-loop system (1.6) is negative which proves the robustness of the method.

Note that the feedback law (1.7) depends on the state $\chi = (\rho, \beta_1, \ldots, \beta_\tau)$. However, we do not always have access to the initial state ρ_0. When the measurements are perfect and we do not know the initial state, the natural estimation of the state consists of taking an arbitrary initial estimate state ρ_0^e and leaving it evolve according to the jump dynamics induced by the measurement results. So the dynamics becomes for the estimate state as follows

$$\begin{cases} \rho_{k+1}^e &= \mathbb{M}_{\mu_k}(\mathbb{D}_{\beta_{\tau,k}}(\rho_k^e)) \\ \beta_{1,k+1} &= \alpha_k \\ \beta_{2,k+1} &= \beta_{1,k} \\ &\vdots \\ \beta_{\tau,k+1} &= \beta_{\tau-1,k}. \end{cases} \tag{1.8}$$

We consider now the joint system-observer dynamics defined for the state

$$\Xi_k = (\rho_k, \rho_k^e, \beta_{1,k}, \ldots, \beta_{\tau,k}):$$

$$\begin{cases} \rho_{k+1} & = \mathbb{M}_{\mu_k}(\mathbb{D}_{\beta_{\tau,k}}(\rho_k)) \\ \rho^e_{k+1} & = \mathbb{M}_{\mu_k}(\mathbb{D}_{\beta_{\tau,k}}(\rho^e_k)) \\ \beta_{1,k+1} & = \alpha_k \\ \beta_{2,k+1} & = \beta_{1,k} \\ & \vdots \\ \beta_{\tau,k+1} & = \beta_{\tau-1,k}. \end{cases} \tag{1.9}$$

We prove the following theorem showing the convergence of the filter ρ^e_k and the physical state ρ_k towards the target state $\bar{\rho}$, under an additional assumption ensuring a quantum separation principle (see also [21]).

Theorem 1.4 (Theorem 3.3 (see page 72))**.** *Consider any closed-loop system of the form (1.9), where the feedback law α_k is a function of the quantum filter: $\alpha_k = g(\rho^e_k, \beta_{1,k}, \dots, \beta_{\tau,k})$. Assume moreover that, whenever $\rho^e_0 = \rho_0$ (so that the quantum filter coincides with the closed-loop dynamics (1.6)), the closed-loop system ρ_k and estimate state ρ^e_k converge almost surely towards a fixed pure state $\bar{\rho}$. Then, for any choice of the initial state ρ^e_0, such that $ker\rho^e_0 \subset ker\rho_0$, the trajectories of the system ρ_k and also the estimate state ρ^e_k converge almost surely towards the same pure state: $\rho_k \to \bar{\rho}$ and $\rho^e_k \to \bar{\rho}$.*

Moreover, we linearize the dynamics (1.9) around the target state $\bar{\Xi} = (\bar{\rho}, \bar{\rho}, 0, \dots, 0)$, and in Proposition 3.3 (page 75), we express that the largest Lyapunov exponent of the linearized closed-loop dynamics is negative.

Extensions to discrete-time systems subject to QND measurements are presented in next subsection.

1.2.3 Feedback stabilization under discrete-time QND measurements

In this section, we consider two types of dynamics. The first one corresponds to non-linear Markov chains whose structure is similar to (1.2), i.e., the control operator and the measurement operator are separated. In the aim of stabilizing some predetermined pure states, we have designed a strict control-Lyapunov function. This work has been published in [5].

The second type of dynamics is described by a non-separable Markov chain, i.e., the measurement operators depend on the control inputs u, this type of measurements are also called adaptive measurements (see e.g., [108]). We stabilized these non-linear Markov chains towards some fixed pure states in presence of measurement imperfections and delays. This work is subject to a preprint [6]. This feedback has been experimentally tested at LKB [87, 88].

Design of strict control-Lyapunov function for separable Markov models

In [5], we consider a finite-dimensional quantum system belonging to the Hilbert space $\mathcal{H} = \mathbb{C}^d$ which is being measured (for the photon box, we have $d = n^{\max} + 1$) through a generalized measurement procedure. Between two measurements, the system undergoes a unitary evolution depending on the scalar control input $u \in \mathbb{R}$. The system state is described by the density operator ρ belonging to $\mathcal{D}$, defined as follows:

$$\mathcal{D} := \{\rho \in \mathbb{C}^{d\times d} \mid \quad \rho = \rho^\dagger, \quad \mathrm{Tr}\,(\rho) = 1, \quad \rho \geq 0\}.$$

The evolution of the system is described by the following non-linear Markov chain:

$$\rho_{k+1} = \mathbb{U}_{u_k}(\mathbb{M}_{\mu_k}(\rho_k)). \tag{1.10}$$

Here

- $u_k \in \mathbb{R}$ is the control at step k;
- $\mathbb{U}_u$ is the superoperator

  $$\mathbb{U}_u : \quad \mathcal{D} \ni \rho \mapsto U_u \rho U_u^\dagger \in \mathcal{D},$$

  with the control operator $U_u = \exp(-iuH)$, where $H \in \mathbb{C}^{d\times d}$ denotes the Hamiltonian which is a Hermitian operator $H^\dagger = H$ (for the photon box, we have $H = i(\mathbf{a}^\dagger - \mathbf{a})$);
- μ_k is a random variable taking values $\mu \in \{1, \ldots, m\}$ (with $m > 0$) with probabilities $\mathrm{Tr}\left(M_\mu \rho_k M_\mu^\dagger\right)$ (for the photon box, we have $\mu_k \in \{g, e\}$);
- for each μ, $\mathbb{M}_\mu$ is the superoperator

  $$\mathbb{M}_\mu : \quad \rho \mapsto \tfrac{M_\mu \rho M_\mu^\dagger}{\mathrm{Tr}\left(M_\mu \rho M_\mu^\dagger\right)} \in \mathcal{D}$$

  defined for $\rho \in \mathcal{D}$ such that $\mathrm{Tr}\left(M_\mu \rho M_\mu^\dagger\right) \neq 0$.
- the Kraus operators $M_\mu \in \mathbb{C}^{d\times d}$ satisfy $\sum_{\mu=1}^m {M_\mu}^\dagger M_\mu = \mathbb{1}$.

The goal is to design a feedback law that globally stabilizes the Markov chain (1.10) towards a chosen target state $|\bar{n}\rangle\langle\bar{n}|$, for some $\bar{n} \in \{1, \ldots, d\}$. We suppose also two following crucial assumptions.

Assumption 1.2 (Assumption 4.1 (see page 83))**.** *The measurement operators M_μ are diagonal in the same orthonormal basis $\{\,|n\rangle \mid \quad n \in \{1, \ldots, d\}\}$, therefore $M_\mu = \sum_{n=1}^d c_{\mu,n} |n\rangle\langle n|$ with $c_{\mu,n} \in \mathbb{C}$.*

Assumption 1.3 (Assumption 4.2 (see page 83))**.** *For all $n_1 \neq n_2$ in $\{1, \ldots, d\}$, there exists a $\mu \in \{1, \ldots, m\}$ such that $|c_{\mu,n_1}|^2 \neq |c_{\mu,n_2}|^2$.*

In [3, 36, 68] the feedback laws were obtained by maximizing the fidelity, $\mathrm{Tr}\,(\rho\,|\bar{n}\rangle\langle\bar{n}|)$, with respect to the target state at each time-step. This means that the feedback strategy was based on the same Lyapunov function given by the fidelity between the current and the target state. In [5], we propose a systematic and explicit method to design a new family of control-Lyapunov functions. The main interest of these new Lyapunov functions relies on the crucial fact that whenever ρ_k does not coincide with the target state, the expectation value of such Lyapunov functions increases by a strictly positive value from step k to $k+1$. In closed-loop, these Lyapunov functions become strict and the convergence analysis is notably simplified since the application of Kushner's invariance principles [55] is not necessary. The construction of these strict Lyapunov functions is based on the Hamiltonian H underlying the controlled unitary evolution and relies on the connectivity of the graph attached to H. They are obtained by inverting a Laplacian matrix derived from H and the quantum states that are stationary under the QND measurements.

We prove the following theorem. It gives a feedback law stabilizing the Markov chain (1.10) towards the pure state $\bar{\rho} = |\bar{n}\rangle\langle\bar{n}|$.

Theorem 1.5 (Theorem 4.2 (see page 88))**.** *Consider the controlled Markov chain of state ρ_k obeying (1.10). Take $\bar{n} \in \{1, \ldots, d\}$ and assume that the graph G^H associated to the Hamiltonian H is connected and that the Kraus operators satisfy Assumptions 1.2 and 1.3. Then there exists a vector $\sigma = (\sigma_n)_{n\in\{1,\ldots,d\}}$ of $\mathbb{R}^d$ such that $-R^H\sigma = \lambda$, where λ is a vector of $\mathbb{R}^d$ of components $\lambda_n < 0$ for $n \neq \bar{n}$ and $\lambda_{\bar{n}} = -\sum_{n\neq\bar{n}} \lambda_n$, and R^H is a Laplacian matrix defined in (4.6) (see Lemma 4.1 (page 86)). Denote by $\rho_{k+\frac{1}{2}} = \mathbb{M}_{\mu_k}(\rho_k)$ the quantum state just after the measurement outcome μ_k at step k. Take $\bar{u} > 0$ and consider the following feedback law*

$$u_k = K(\rho_{k+\frac{1}{2}}) = \underset{u\in[-\bar{u},\bar{u}]}{argmin}\Big(V_\epsilon\big(\mathbb{U}_u(\rho_{k+\frac{1}{2}})\big)\Big), \tag{1.11}$$

where the control-Lyapunov function $V_\epsilon(\rho)$ is defined by

$$V_\epsilon(\rho) = \sum_{n=1}^{d}\left(\sigma_n \langle n|\rho|n\rangle - \tfrac{\epsilon}{2}\left(\langle n|\rho|n\rangle\right)^2\right) \tag{1.12}$$

with the parameter $\epsilon > 0$ not too large to ensure that

$$\forall n \in \{1, \ldots, d\} \setminus \{\bar{n}\},\ \ \lambda_n + 2\epsilon\left((\langle n|H|n\rangle)^2 - \langle n|H^2|n\rangle\right) > 0.$$

Then for any $\rho_0 \in \mathcal{D}$, the closed-loop trajectory ρ_k converges almost surely to the pure state $|\bar{n}\rangle\langle\bar{n}|$.

When the measurement process is perfect, the Markov process (1.10) represents a natural choice for estimating the hidden state ρ. Thus, the estimate ρ^e of ρ satisfies the following dynamics

$$\rho^e_{k+1} = \mathbb{U}_{u_k}(\mathbb{M}_{\mu_k}(\rho^e_k)), \tag{1.13}$$

where the measurement outcome μ_k is driven by (1.10). We prove the following theorem which is the analogue of Theorem 1.4.

Theorem 1.6 (Theorem 4.3 (see page 90))**.** *Consider the Markov chain of state ρ_k obeying (1.10) and assume that the assumptions of Theorem 1.5 are satisfied. For each measurement outcome μ_k given by (1.10), consider the estimate ρ^e_k given by (1.13) with an initial condition ρ^e_0.*

Set $u_k = K(\rho^e_{k+\frac{1}{2}})$*, where* K *is given by* (1.11)*. Then for all* $\epsilon \in \left]0, \min_{n\neq\bar{n}}\left(\frac{\lambda_n}{R^H_{n,n}}\right)\right]$ *(where the Laplacian matrix* R^H *is defined in Equation* (4.6) *(see page 86)),* ρ_k *and* ρ^e_k *converge almost surely towards the target state* $|\bar{n}\rangle\langle\bar{n}|$ *as soon as* $\ker(\rho^e_0) \subset \ker(\rho_0)$*.*

Feedback design for non-separable Markov models including delays and imperfections

In [6], we consider a finite-dimensional quantum system belonging to the Hilbert space $\mathcal{H} = \mathbb{C}^d$, which is being measured by a discrete-time sequence of Positive Operator Valued Measures (POVMs), and which is not fixed but is a function of some classical control signal u, similarly to [108]. The random evolution of the state $\rho_k \in \mathcal{D}$ is given through the following non-linear Markov chain:

$$\rho_{k+1} = \mathbb{M}^{u_{k-\tau}}_{\mu_k}(\rho_k), \tag{1.14}$$

where

- $u_{k-\tau}$ is the control at step k, subject to a delay of $\tau > 0$ steps. This delay is usually due to delays in the measurement process that can also be seen as delays in the control process,
- μ_k is a random variable taking values μ in $\{1, \dots, m\}$ with probability
$$p^{u_{k-\tau}}_{\mu,\rho_k} = \mathrm{Tr}\left(M^{u_{k-\tau}}_\mu \rho_k M^{u_{k-\tau}\dagger}_\mu\right),$$
- for each μ, $\mathbb{M}^u_\mu$ is the superoperator
$$\mathbb{M}^u_\mu : \quad \rho \mapsto \frac{M^u_\mu \rho M^{u\dagger}_\mu}{\mathrm{Tr}\left(M^u_\mu \rho M^{u\dagger}_\mu\right)} \in \mathcal{D} \tag{1.15}$$
defined for $\rho \in \mathcal{D}$ such that $p^u_{\mu,\rho} = \mathrm{Tr}\left(M^u_\mu \rho M^{u\dagger}_\mu\right) \neq 0$.

In addition, for each u, the Kraus operators $M^u_\mu = \mathbb{C}^{d\times d}$ satisfy $\sum_{\mu=1}^m M^{u\dagger}_\mu M^u_\mu = \mathbb{1}$. We assume that the Kraus operators M^u_μ cannot be decomposed as follows: $M^u_\mu = U_u M^0_\mu$, with $U_0 = \mathbb{1}$.

Now we define the Kraus map $\mathbb{K}^u$.

Definition 1.2. *The Kraus map* $\mathbb{K}^u$ *is defined as follows*

$$\mathcal{D} \ni \rho \mapsto \mathbb{K}^u(\rho) = \sum_{\mu=1}^m M^u_\mu \rho M^{u\dagger}_\mu \in \mathcal{D}.$$

Our aim is to design a feedback law that globally stabilizes the Markov chain (1.14) towards a chosen target state $|\bar{n}\rangle\langle\bar{n}|$, for some $\bar{n} \in \{1, \dots, d\}$. We suppose the following assumptions.

Assumption 1.4 (Assumption 4.3 (see page 91))**.** *For $u = 0$, the measurement operators M_μ^0 are diagonal in the same orthonormal basis $\{\,|n\rangle \mid n \in \{1, \ldots, d\}\}$, therefore $M_\mu^0 = \sum_{n=1}^{d} c_{\mu,n} |n\rangle \langle n|$ with $c_{\mu,n} \in \mathbb{C}$.*

Assumption 1.5 (Assumption 4.4 (see page 91))**.** *For all $n_1 \neq n_2$ in $\{1, \ldots, d\}$, there exists $\mu \in \{1, \ldots, m\}$ such that $|c_{\mu,n_1}|^2 \neq |c_{\mu,n_2}|^2$.*

Assumption 1.6 (Assumption 4.5 (see page 91))**.** *The measurement operators M_μ^u are C^2 functions of u.*

In [6], we propose a stabilizing feedback law based on a set of Lyapunov functions made of linear combinations of the martingales attached to these QND measurements. These Lyapunov functions define a "distance" between the target state and the current one. The parameters of these Lyapunov functions are given by inverting Metzler matrices characterizing the impact of the control input on the Kraus operators defining the Markov processes and POVMs. The (graph theoretic) properties of the Metzler matrices are used to construct families of open-loop supermartingales that become strict supermartingales in closed-loops. This fact provides directly the convergence to the target state without using Kushner's invariance principle in contrast to the proof given in previous works [3, 68].

As explained in previous subsection, in order to take the delays into account, we can take $(\rho_k, u_{k-1}, \ldots, u_{k-\tau})$ as the state at step k. More precisely, we consider the state $\chi = (\rho, \beta_1, \ldots, \beta_\tau)$, where β_l stands for the control input u delayed l steps. Then the state form of the delay dynamics (1.14) is governed by the following Markov chain

$$\begin{cases} \rho_{k+1} &= \mathbb{M}_{\mu_k}^{\beta_{\tau,k}}(\rho_k) \\ \beta_{1,k+1} &= u_k \\ \beta_{2,k+1} &= \beta_{1,k} \\ & \vdots \\ \beta_{\tau,k+1} &= \beta_{\tau-1,k}. \end{cases} \tag{1.16}$$

We can now state our main result.

Theorem 1.7 (Theorem 4.7 (see page 97))**.** *Consider the Markov chain (1.16) with Assumptions 1.4, 1.5 and 1.6. Take $\bar{n} \in \{1, \ldots, d\}$ and assume that the directed graph G associated to the Metzler matrix R defined in Lemma 4.3 (page 93) is strongly connected. Take $\epsilon > 0$, $\sigma \in \mathbb{R}_+^d$ the solution of $R\sigma = \lambda$ with $\sigma_{\bar{n}} = 0$, $\lambda_n < 0$ for $n \in \{1, \ldots, d\} \setminus \{\bar{n}\}$, $\lambda_{\bar{n}} = -\sum_{n \neq \bar{n}} e_n \lambda_n$ (see Lemma 4.4 (page 93)) and consider*

$$V_\epsilon(\rho) = \sum_{n=1}^{d} \sigma_n \langle n| \rho |n\rangle - \tfrac{\epsilon}{2}(\langle n| \rho |n\rangle)^2.$$

Take $\bar{u} > 0$ and consider the following feedback law

$$u_k = \underset{\xi \in [-\bar{u}, \bar{u}]}{\mathit{argmin}} \Big(\mathbb{E}\left[W_\epsilon(\chi_{k+1}) | \chi_k, u_k = \xi \right] \Big) =: \mho(\chi_k), \tag{1.17}$$

where $W_\epsilon(\chi) = V_\epsilon(\mathbb{K}^{\beta_1}(\mathbb{K}^{\beta_2}(\ldots\ldots\mathbb{K}^{\beta_\tau}(\rho)\ldots)))$.
Then, there exists $u^* > 0$ *such that for all* $\bar{u} \in]0, u^*]$ *and* $\epsilon \in \left]0, \min_{n\neq\bar{n}}\left(\frac{\lambda_n}{R_{n,n}}\right)\right]$*, the closed-loop Markov chain of state* χ_k *with the feedback law* (1.17) *converges almost surely towards* $(|\bar{n}\rangle\langle\bar{n}|, 0, \ldots, 0)$ *for any initial condition* $\chi_0 \in \mathcal{D} \times [-\bar{u}, \bar{u}]^\tau$.

When the measurements are perfect, a natural state estimate ρ^e follows the following dynamics
$$\rho^e_{k+1} = \mathbb{M}^{u_{k-\tau}}_{\mu_k}(\rho^e_k) \tag{1.18}$$
where the measurement outcome μ_k is driven by (1.14).

We now state the following separation principle type theorem which ensures the convergence of the physical state ρ and estimated state ρ^e.

Theorem 1.8 (Theorem 4.8 (see page 102))**.** *Consider the Markov chain of state* ρ_k *obeying* (1.14) *and assume that the assumptions of Theorem 1.7 are satisfied. For each measurement outcome* μ_k *given by* (1.14)*, consider the estimation* ρ^e_k *with an initial condition* ρ^e_0.
Set $u_k = \mho(\rho^e_k, u_{k-1}, \ldots, u_{k-\tau})$*, where* $\mho$ *is given by* (1.17)*. Then, there exists* $u^* > 0$ *such that for all* $\bar{u} \in]0, u^*]$ *and* $\epsilon \in \left]0, \min_{n\neq\bar{n}}\left(\frac{\lambda_n}{R_{n,n}}\right)\right]$*,* ρ_k *and* ρ^e_k *converge almost surely towards the target state* $|\bar{n}\rangle\langle\bar{n}|$ *as soon as* $\ker(\rho^e_0) \subset \ker(\rho_0)$.

In Theorem 4.9 (page 104), we prove the convergence of a feedback law which takes into account the measurement errors and delays. It relies on measurements that can be corrupted by random errors with conditional probabilities described by a known left stochastic matrix. This imperfect measurement model is introduced in [90]. [4] Also, we will recover the quantum separation principle in Theorem 4.10 (page 104). This separation principle implies that the feedback may be designed only knowing the state of the quantum filter and measurement outcomes in spite of not knowing the actual state of the system due to the imperfections in measurement.

Closed-loop simulations (Figures 4.3 (page 111) and 4.4 (page 113)) corroborated by experimental data illustrate the interest of such nonlinear feedback scheme for the photon box.

1.2.4 Stability of continuous-time quantum filters

The lack of knowledge about the initial state of the system lead to use quantum filtering theory to estimate the real state of the system. The key issues are concerned with the stability and the convergence of the quantum filter. Few convergence results are available up to now, except the sufficient conditions established in [101]. We focus here on stability issues.

Indeed, the convergence means the independence of the filter, after a long time, from the initial state estimate. In [101], necessary conditions for asymptotic stability of the

[4] See Theorem 2.2 (page 47).

quantum filter is given. They are related to observability of the system. The observability means that there do not exist two different initial states which give rise to measurement outcomes with the same probability. The stability of the quantum filter is a problem which can be defined in discrete-time as well as in continuous-time.

Similarly to the last subsection, we can consider the following discrete-time dynamics together with filter:

$$\rho_{k+1} = \frac{M_{\mu_k}\rho_k M_{\mu_k}^\dagger}{\mathrm{Tr}\left(M_{\mu_k}\rho_k M_{\mu_k}^\dagger\right)}, \tag{1.19}$$

$$\rho_{k+1}^e = \frac{M_{\mu_k}\rho_k^e M_{\mu_k}^\dagger}{\mathrm{Tr}\left(M_{\mu_k}\rho_k^e M_{\mu_k}^\dagger\right)}, \tag{1.20}$$

where μ_k is a random variable which takes values $\mu \in \{1, \dots, m\}$, with probability $\mathrm{Tr}\left(M_\mu \rho_k M_\mu^\dagger\right)$.

Stability of the quantum filter can be proven by showing that the distance between the physical state ρ_k and its associated quantum filter ρ_k^e decreases in average. The distance between two density matrices can be defined by one minus the fidelity between these two density matrices. In [75, Chapter 9], the fidelity between two arbitrary density matrices ρ and σ is defined as follows:

$$F(\rho, \sigma) = \mathrm{Tr}^2\left(\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right). \tag{1.21}$$

This is a real number between 0 and 1. Moreover, $F(\rho, \sigma) = 1$ means $\rho = \sigma$, and $F(\rho, \sigma) = 0$ means that the support of ρ and σ are orthogonal. $F(\rho, \sigma)$ coincides with their inner Frobenius product $\mathrm{Tr}\,(\rho\sigma)$ when at least one of the states ρ or σ is pure.

The following theorem, based only on Cauchy-Schwartz inequalities, proves that the Frobenius inner product between the estimate and physical state is always a submartingale whatever the purity of the estimate and physical state are.

Theorem 1.9 (Mirrahimi et al. [68]). *Consider the process* (1.2) *and the associated filter satisfying* $\rho_{k+1}^e = \mathbb{M}_{\mu_k}(\mathbb{D}_{\alpha_k}(\rho_k^e))$, *for any arbitrary control input* $(\alpha_k)_{k=1}^\infty$. *We have* $\mathbb{E}\left[\mathrm{Tr}\,(\rho_k\rho_k^e)\right] \leq \mathbb{E}\left[\mathrm{Tr}\left(\rho_{k+1}\rho_{k+1}^e\right)\right],\ \forall k.$

Now we announce the following theorem which generalizes Theorem 1.9 by proving that the fidelity defined in (1.21) is a sub-martingale for any arbitrary purity of the states.

Theorem 1.10 (Rouchon [82]). *Consider the Markov chain of state* (ρ_k, ρ_k^e) *satisfying* (1.19) *and* (1.20). *Then the fidelity* $F(\rho_k, \rho_k^e)$ *defined in* (1.21) *is a submartingale:*

$$\mathbb{E}\left[F(\rho_{k+1}, \rho_{k+1}^e) | (\rho_k, \rho_k^e)\right] \geq F(\rho_k, \rho_k^e).$$

In [33], the convergence of the continuous-time estimate to the physical state is discussed for a generic Hamiltonian and measurement operator. The analysis relies on the fact that, for pure states, the fidelity between the real state and its estimate is proved to be a submartingale. Also, the convergence issues are discussed.

In [2], we have extended this result by providing the stability of the quantum filter attached to a stochastic master equation driven only by a Wiener process. In fact, we show that for any arbitrary purity, the fidelity defined in (1.21) is a submartingale.

Let us now state more precisely this problem. We consider quantum systems of finite dimensions $1 < d < \infty$. If ρ_t designs the state of the system at time t, the time-evolution characterized by a diffusive equation has the following form (cf., [16, 20, 103])

$$d\rho_t = \mathcal{L}(\rho_t)\,dt + \left(L\rho_t + \rho_t L^\dagger - \mathrm{Tr}\left((L+L^\dagger)\rho_t\right)\rho_t\right) dW_t\ , \tag{1.22}$$

where

- dW_t is the Wiener process which is the following innovation

$$dW_t = dy_t - \ \mathrm{Tr}\left((L+L^\dagger)\,\rho_t\right)\ dt\ , \tag{1.23}$$

  where y_t is a continuous semi-martingale with quadratic variation $\langle y, y\rangle_t = t$ (which is the observation process obtained from the system) and L is an arbitrary matrix which determines the measurement process (typically the coupling to the probe field for quantum optics systems) ;

- the superoperator $\mathcal{L}$ is the Lindblad operator given by

$$\mathcal{L}(\rho) := -i[H,\rho] - \frac{1}{2}\{L^\dagger L, \rho\} + L\rho L^\dagger,$$

  where $H = H^\dagger$ is a Hermitian operator. [5]

The evolution of the quantum filter of state ρ_t^e associated to (1.22) is described by the following stochastic master equation. It depends on the continuous-time measurement y_t which is function of the true quantum state ρ_t via (1.23) (see e.g., [7]):

$$d\rho_t^e = \mathcal{L}(\rho_t^e)\,dt + \left(L\rho_t^e + \rho_t^e L^\dagger - \mathrm{Tr}\left((L+L^\dagger)\rho_t^e\right)\rho_t^e\right)\left(dy_t - \ \mathrm{Tr}\left((L+L^\dagger)\rho_t^e\right)dt\right). \tag{1.24}$$

We state now our main result.

Theorem 1.11 (Theorem 5.1 (see page 121))**.** *Consider the Markov processes (ρ_t, ρ_t^e) satisfying stochastic master Equations (1.22) and (1.24) respectively with ρ_0, ρ_0^e in $\mathcal{D}$. Then the fidelity $F(\rho_t, \rho_t^e)$, defined in Equation (1.21), is a submartingale, i.e.,*

$$\mathbb{E}\left[F(\rho_t,\rho_t^e)|(\rho_s,\rho_s^e)\right] \geq F(\rho_s,\rho_s^e), \quad \forall t \geq s.$$

Roughly speaking, this result implies the stability of the quantum filter ρ_t^e. Indeed, we show that the distance between the real state and its associated quantum filter decreases in average. This means that if $F(\rho_0, \rho_0^e)$ is near one, then in average, the fidelity $F(\rho_t, \rho_t^e)$ is closer to one, since $\mathbb{E}\left[F(\rho_t, \rho_t^e)\right] \geq F(\rho_0, \rho_0^e)$.

[5] The notation $\{A, B\}$ refers to $AB + BA$

Stability and filter design in presence of imperfections

Another type of stochastic master equation is driven by a Poisson process:

$$d\rho_t = \mathcal{L}(\rho_t)\, dt + \left(\frac{C\rho_t C^\dagger}{\mathrm{Tr}\left(C\rho_t C^\dagger\right)} - \rho_t \right) \left(dN_t - \mathrm{Tr}\left(C\rho_t C^\dagger\right)\, dt \right) .$$

Here

- dN_t is the following process $dN_t = N(t+dt) - N(t)$, where $N(t)$ is a Poisson process taking values one with probability $\mathrm{Tr}\left(C\rho_t C^\dagger\right) dt$ and zero with the complementary probability;
- the superoperator $\mathcal{L}$ is the Lindblad operator defined by,

$$\mathcal{L}(\rho) := -i[H, \rho] - \frac{1}{2}\{C^\dagger C, \rho\} + C\rho C^\dagger;$$

- and C is an arbitrary matrix.

More general stochastic master equations are called jump-diffusion stochastic master equations which are driven by both multidimensional Poisson and Wiener processes. In Chapter 6, through some heuristic arguments, we will obtain a generic form of the jump-diffusion stochastic master equations, in presence of classical measurement imperfections. To our knowledge, this problem has not been treated elsewhere. Our heuristic result, Equation (6.16) (page 141) is an extension of [90, Theorem III.1] to continuous-time. The jump-diffusion stochastic master equation of the estimated state, in presence of measurement errors will be given in Equation (6.19) (page 141)). Also, we claim the stability of such generalized estimate optimal filter in Conjecture 1 (page 142).

We believe that it is possible to extend the approach of [77], in order to make these heuristic arguments more rigorous. This will be done in a future work.

1.3 Outline

Here is a brief description of the contents of this thesis.

Chapter 2 contains an introduction to discrete-time open quantum systems. The aim of this chapter is to provide most of the necessary quantum notations and the essential theorems that we need for the remaining chapters.

Chapter 3 is concerned with the feedback scheme which stabilizes an arbitrary photon-number state in a microwave cavity. The quantum non-demolition (QND) measurement of the cavity state allows a non-deterministic preparation of Fock states. Here, by the mean of a controlled field injection, we desire to make this preparation process deterministic. The system evolves through a discrete-time Markov process and we design the feedback law applying Lyapunov techniques. Also, in our feedback design we take into account an unavoidable pure delay and we compensate it by a stochastic version of a Kalman-type predictor. After illustrating the efficiency of the proposed feedback law through simulations, we provide a rigorous proof of the global stability of the closed-loop system based on tools from stochastic stability analysis. A brief study of the Lyapunov exponents of the linearized system around the target state gives a strong indication of the robustness of the method. The results of this chapter are based on [3].

Chapter 4 develops the mathematical methods underlying a recent measurement-based feedback experiment [88] stabilizing photon-number states. It considers a controlled system whose quantum state, a finite-dimensional density operator, is governed by a discrete-time nonlinear Markov process. This chapter generalizes the previous chapter to any discrete-time quantum systems under QND measurements. This generalization will be done in three steps: first, when the measurement operator and the control operator are separable; second, when the measurement operator and the control operator are non-separable (the measurement operator depend on the control input); third, when the imperfections and delays are added. Indeed, we design the new families of Lyapunov functions for each Markov model which facilitate the convergence proof. Also, the feedback laws obtained by these Lyapunov functions are more efficient in practice.

Closed-loop simulations and experimental data [88, 87] confirm the efficiency of such nonlinear feedback scheme for the photon box. The results of this chapter are based on [5, 6, 88].

Chapter 5 proves that the fidelity between the quantum state governed by a continuous-time stochastic master equation driven by a Wiener process and its associated quantum-filter state is a submartingale. This result is a generalization to non-pure quantum states where fidelity does not coincide in general with a simple Frobenius inner product. This result implies the stability of such filtering process but does not necessarily ensure its asymptotic convergence. This chapter results from our work published in [2].

Chapter 6 presents more general stochastic master equations which are driven by multi-dimensional Poisson or/and Wiener processes. Heuristically, we obtain the form of stochastic master equations driven by both multidimensional Poisson and Wiener processes when the measurement processes are not perfect. We conjecture the stability of such optimal filter.

Conclusion proposes some natural extensions to the results described in the chapters mentioned above.

Appendix A provides the different notions of stability and some stability theorems for stochastic processes.

Part I

Discrete-time open quantum system

Chapter 2

Models of discrete-time open quantum systems

Contents

Pour un système quantique, être ouvert signifie être en interaction avec son environnement. En particulier, la mesure d'un observable physique requiert une interaction entre un instrument de mesure et le système. Cette interaction modifie l'état du système quantique (effet secondaire de la mesure). En effet, la récupération d'une information sur l'état quantique est uniquement possible à condition d'ouvrir le système sur son environnement.

Dans ce chapitre, nous introduisons brièvement les notations nécessaires et les types de mesures que nous utiliserons dans les prochains chapitres. Nous décrivons également le processus stochastique derrière des opérateurs positifs de mesure (POVM) utilisé pour les mesures parfaites et nous rappelons les résultats de stabilité du filtre quantique en temps discret. De plus, nous présentons le modèle de mesures imparfaites que nous considérons dans cette thèse. Les propriétés de stabilité de la procédure d'estimation seront aussi rappelées dans ce cadre. Notre présentation s'inspire beaucoup de [44], [112], [92] et [69].

In the context of quantum systems, being open stands for being in interaction with its environment. In particular, measuring a physical observable necessitates the interaction of a meter with the system and leads to modification of the state of the quantum system (measurement back-action). Indeed, retrieving any information on the quantum state comes at the expense of opening the system to its environment.

In this chapter, we briefly introduce all the necessary notations and different types of measurements that we need in the following chapters. Also we describe the stochastic process attached to positive operator valued measure (POVM) for the perfect measurements and we will recall the stability results of the discrete-time quantum filter. Moreover, we present the imperfect measurement model that we consider in this thesis. In addition, the stability results for the estimation procedure will be reminded in this case. Our presentation is very much inspired from [44], [112], [92] and [69].

2.1 Bra and Ket

All the objects described in this chapter are mathematically well-defined in a finite dimensional Hilbert space $\mathcal{H}$. When the Hilbert spaces are of infinite dimension, one can still define these objects but one needs the mathematical justifications depending on the considered system. We suppose in the following that the Hilbert spaces under consideration are always of finite dimension.

Ket $|.\rangle$ denotes a vector in the Hilbert space $\mathcal{H}$ while Bra $\langle.|$ denotes a co-vector in the dual of the Hilbert space $\mathcal{H}$. The quantum state can be represented by a vector $|\psi\rangle$ (wave function) in the Hilbert space $\mathcal{H}$, with $\langle\psi|\psi\rangle = \| \, |\psi\rangle \, \|^2 = 1$.

2.2 Composite system

A composite system is made of several sub-systems. The state of a composite physical system is the tensor product of the state spaces of its component physical sub-systems. Consider now a system composed of two sub-systems A and B with corresponding state spaces $\left|\psi^A\right\rangle$ and $\left|\psi^B\right\rangle$ where these states are represented in the Hilbert spaces $\mathcal{H}^A$ and $\mathcal{H}^B$. The joint state of the total system is $\left|\psi^A\right\rangle \otimes \left|\psi^B\right\rangle$ living in the tensor product Hilbert space $\mathcal{H}^{AB} = \mathcal{H}^A \otimes \mathcal{H}^B$. Notice that most of the unitary vectors $\left|\psi^{AB}\right\rangle$ in $\mathcal{H}^{AB}$ are not the tensor product of two unitary vectors of $\mathcal{H}^A$ and $\mathcal{H}^B$. In this case $\left|\psi^{AB}\right\rangle$ represents an entangled state between sub-systems A and B. Finally, notice that contrarily to classical composite systems, the state space is not obtained by a Cartesian product but by a tensor product.

2.3 Schrödinger equation

Consider now a time dependant system whose state at time t is described by the state $|\psi_t\rangle$ which evolves according to the Schrödinger equation (here we take the Planck constant $\hbar = 1$):

$$i\frac{d}{dt}|\psi_t\rangle = H(t)|\psi_t\rangle,$$

where $i^2 = -1$ and H is a time-varying Hermitian operator, called the Hamiltonian. Note that this means that $H^\dagger = H$, where $H^\dagger$ is the adjoint operator. If the initial state is $|\psi_0\rangle$, then the state $|\psi_t\rangle$ of the system at time t is given by the following equation

$$|\psi_t\rangle = U_t|\psi_0\rangle,$$

with the linear operator U_t satisfying the following dynamics

$$i\frac{d}{dt}U_t = H(t)U_t, \quad U_0 = \mathbb{1}.$$

The fact that H is Hermitian implies that U_t is a unitary operator, i.e., $U_t^\dagger U_t = U_t U_t^\dagger = \mathbb{1}$. Therefore, the evolution does not change the norm of the state vector, which was set equal to one. The operator U_t is called the propagator.

2.4 Density operator

The unit vector $|\psi\rangle$ in the Hilbert space $\mathcal{H}$ corresponds to a pure state. It is associated to the rank one projector $\rho = |\psi\rangle\langle\psi|$ which satisfies $\mathrm{Tr}(\rho) = \mathrm{Tr}(\rho^2) = 1$. In general, the state of a quantum system is not pure and can be a statistical mixture of orthogonal pure states $|\psi_k\rangle$ living in $\mathcal{H}$. In this case, the analogue of the rank one projector ρ, the density operator, reads

$$\rho = \sum_\nu p_\nu |\psi_\nu\rangle\langle\psi_\nu|,$$

where p_ν is the probability that system is in the pure state $|\psi_\nu\rangle$. We still have $\mathrm{Tr}(\rho) = \sum_\nu p_\nu = 1$. One says that ρ is a mixed state if $\mathrm{Tr}(\rho^2) < 1$. In this case the density operator ρ represents the quantum state: it summarizes an observer's knowledge on the system. Any operator ρ on $\mathcal{H}$ that is Hermitian, semi-definite positive and of trace one is a density operator. The above decomposition corresponds then to its spectral decomposition with the normalized eigenstate $|\psi_\nu\rangle$ associated to the eigenvalue p_ν.

For a composite system $\mathcal{H}^{AB} = \mathcal{H}^A \otimes \mathcal{H}^B$, its quantum state is described by the density operator ρ^{AB} that is not in general the tensor product of a density operator σ on $\mathcal{H}^A$

and ς on $\mathcal{H}^B$. Nevertheless, one can define the partial trace with respect to B as the linear map sending ρ^{AB} to a density operator ρ^A on $\mathcal{H}^A$, $\rho^A = \mathrm{Tr}_B\left(\rho^{AB}\right)$. This linear map is completely defined by the fact that if $\rho^{AB} = \sigma \otimes \varsigma$ is a separable state, then $\rho^A = \mathrm{Tr}_B\left(\sigma \otimes \varsigma\right) = \sigma \mathrm{Tr}\left(\varsigma\right) = \sigma$.

2.5 Observables and measurements

Measurements are associated to the observables corresponding to Hermitian operators on the Hilbert space $\mathcal{H}$. In fact an observable has the following spectral decomposition

$$\mathcal{O} = \sum_\mu \lambda_\mu P_\mu,$$

where λ_μ are the eigenvalues of the Hermitian operator $\mathcal{O}$ which are real, and P_μ are the orthogonal projectors on the corresponding eigenspaces.

2.5.1 Projective measurement

When measuring the observable $\mathcal{O}$, the result of the measurement will necessarily be one of the eigenvalues of $\mathcal{O}$. Achieving the measurement result λ_μ, the quantum state ρ just after this measurement becomes ρ_+ which is given by

$$\rho_+ = \frac{P_\mu \rho P_\mu}{p_\mu}.$$

Here $p_\mu = \mathrm{Tr}\left(P_\mu \rho\right)$ is the probability to achieve the measurement result λ_μ. (Note that $P_\mu = P_\mu^\dagger$ and $P_\mu^2 = P_\mu$.) In addition, the average value of $\mathcal{O}$ is given by $\sum_\mu p_\mu \lambda_\mu = \mathrm{Tr}\left(\rho \mathcal{O}\right)$.

Repeating after this first measurement a second one, one gets with probability one λ_μ. This results from the fact that $\mathrm{Tr}\left(P_\nu \rho_+\right)$ is equal to 0 if $\nu \neq \mu$ and is equal to 1 if $\nu = \mu$. Moreover, this second measurement leaves ρ_+ unchanged since $P_\mu \rho_+ P_\mu^\dagger = \rho_+$.

For pure states, we have $\rho = |\psi\rangle\langle\psi|$, and the projective measurement is simply given by

$$|\psi\rangle_+ = \frac{P_\mu |\psi\rangle}{\sqrt{p_\mu}},$$

with $p_\mu = \langle\psi| P_\mu |\psi\rangle$. When the observable $\mathcal{O}$ is non-degenerate, all projectors P_μ are of rank one. Such a projective measurement is called a von Neumann measurement.

2.5.2 Positive operator valued measure (POVM)

In general, we do not directly measure the system of interest. In fact, we consider the interaction of the system with its environment, and the experimenter observes the effect of the system on the environment. In the aim of formulating this scheme, one defines the large Hilbert space $\mathcal{H}^{SM} = \mathcal{H}^S \otimes \mathcal{H}^M$, where $\mathcal{H}^S$ denotes the Hilbert space of the quantum system S and $\mathcal{H}^M$ denotes the Hilbert space of the meter, and one supposes that the initial state (before the measurement) of the system is described by the pure state $|\psi^S\rangle \otimes |\theta\rangle$. The measurement process is described by the unitary evolution U_{SM} on $\mathcal{H}^{SM}$ of the initial state producing, in general, an entangled state $U_{SM}(|\psi^S\rangle \otimes |\theta\rangle)$ after an interaction time of τ between the system and the meter. The meter's observable has the following form: $\mathcal{O}_M = \mathbb{1}_S \otimes \sum_\mu \lambda_\mu P_\mu$. Here P_μ is a rank-one projection, i.e., $P_\mu = |\lambda_\mu\rangle \langle\lambda_\mu|$ with $|\lambda_\mu\rangle \in \mathcal{H}^M$. One can now formulate the measurement procedure as follows:

$$(M_\mu |\psi^S\rangle) \otimes |\lambda_\mu\rangle = (\mathbb{1}_S \otimes P_\mu) U_{SM}(|\psi^S\rangle \otimes |\theta\rangle), \tag{2.1}$$

where the measurement operators M_μ are defined as follows. For all $|\psi^S\rangle \in \mathcal{H}^S$,

$$U_{SM}(|\psi^S\rangle \otimes |\theta\rangle) = \sum_\mu (M_\mu |\psi^S\rangle) \otimes |\lambda_\mu\rangle .$$

After the measurement, the conditional state of the system given the outcome λ_μ is $|\psi^S\rangle_+ = \frac{M_\mu |\psi^S\rangle}{\sqrt{p_\mu}}$. This can be also extended to the case of a mixed state ρ on $\mathcal{H}^S$, where the probability of obtaining the value λ_μ is simply given by $p_\mu = \mathrm{Tr}\left(M_\mu \rho M_\mu^\dagger\right)$, and the conditional state given the outcome λ_μ is

$$\rho_+ = \mathbb{M}_\mu(\rho) := \frac{M_\mu \rho M_\mu^\dagger}{\mathrm{Tr}\left(M_\mu \rho M_\mu^\dagger\right)}.$$

Here $\mathbb{M}_\mu$ is a non-linear superoperator, i.e., it maps an operator to another operator in $\mathcal{H}^S$. From (2.1), it is clear that the measurement operators satisfy $\sum_\mu M_\mu^\dagger M_\mu = \mathbb{1}_S$ (since U_{SM} is unitary). This property and the positiveness of the operators $M_\mu^\dagger M_\mu$ are the necessary conditions for the set $\{M_\mu\}$ to define a positive operator valued measure (POVM).

2.5.3 Quantum non-demolition (QND) measurement

Quantum non-demolition (QND) measurements correspond to POVM measurements with specific propagator U_{SM} respecting a given observable $\mathcal{O}_S$ on system S: if the state ρ of the system S is an eigenstate of $\mathcal{O}_S$, it is unchanged by the POVM described previously. In other words, $\rho_+ = \rho$ for any μ.

The propagator U_{SM} is generated by the total Hamiltonian $H = H_S + H_M + H_{SM}$ on $\mathcal{H}^S \otimes \mathcal{H}^M$. In this equation, H_S is the Hamiltonian of the subsystem S which is of the form $H_S = \widetilde{H}_S \otimes \mathbb{1}_M$, where $\widetilde{H}_S$ is a Hermitian operator on $\mathcal{H}^S$. Similarly, $H_M = \mathbb{1}_S \otimes \widetilde{H}_M$, where $\widetilde{H}_M$ is a Hermitian operator defined on $\mathcal{H}^M$. The operator H_{SM} is the system-meter interaction Hamiltonian acting on $\mathcal{H}^S \otimes \mathcal{H}^M$. Let U_{SM} be the propagator generated by H for an interaction time of τ (for time-invariant H, we have $U_{SM} = e^{-i\tau H}$). To ensure that the POVM measurement is a QND measurement, it is sufficient to have

$$[H, \mathcal{O}_S] = 0.$$

In other words, U_{SM} and $\mathcal{O}_S$ commute. Thus, the eigenstates of $\mathcal{O}_S$ are also the eigenstates of M_μ. Theses eigenstates are unchanged by the POVM measurements.

2.6 Perfect measurement

In this section, we give the stochastic process attached to a POVM and we announce a stability result of its associated quantum filter.

2.6.1 Stochastic process attached to POVM

The set of density operators ρ on the system's Hilbert space $\mathcal{H}^S$ is denoted by $\mathcal{D}$. To any POVM defined by a set of measurement operators (M_μ), is attached the following Markov chain of state space $\mathcal{D}$:

$$\rho_{k+1} = \frac{M_{\mu_k}\rho_k M_{\mu_k}^\dagger}{\mathrm{Tr}\left(M_{\mu_k}\rho_k M_{\mu_k}^\dagger\right)}, \tag{2.2}$$

where ρ_k is the density operator at sampling time $k\tau$, for $k \in \mathbb{N}$, and the probability to have the measurement outcome μ_k equal to $\mu \in \{1, \ldots, m\}$ (with $m \in \mathbb{N}$), at step k, is given by $p_{\mu,\rho_k} = \mathrm{Tr}\left(M_\mu \rho_k M_\mu^\dagger\right)$. Notice that if ρ_k is an element of $\mathcal{D}$, then $\rho_{k+1} \in \mathcal{D}$.

The conditional expectation of the average value of observable $\mathcal{O}$ in $\mathcal{H}^S$ at step $k+1$ knowing the quantum state at step k, ρ_k is given by

$$\mathbb{E}\left[\mathrm{Tr}\left(\mathcal{O}\rho_{k+1}\right)|\rho_k\right] = \mathrm{Tr}\left(\mathcal{O}\mathbb{K}(\rho_k)\right),$$

where $\mathbb{K}$ is the Kraus superoperator attached to the POVM defined by

$$\mathbb{K}(\rho) := \sum_{\mu=1}^{m} M_\mu \rho M_\mu^\dagger.$$

Notice that $\mathbb{K}(\rho_k) = \mathbb{E}\left[\rho_{k+1}|\rho_k\right]$.

Now suppose that this POVM provides a QND measurement of an observable $\mathcal{O}_S$ in $\mathcal{H}^S$ with the spectral decomposition $\mathcal{O}_S = \sum_\mu \lambda_\mu P_\mu$ (λ_μ being the eigenvalues and P_μ corresponding to the eigenspace projectors). Then any P_μ yields to a martingale $\mathrm{Tr}\left(\rho P_\mu\right)$:

$$\mathbb{E}\left[\mathrm{Tr}\left(P_\mu\rho_{k+1}\right)|\rho_k\right] = \mathrm{Tr}\left(P_\mu\rho_k\right).$$

The above equation results from $\sum_{\nu=1}^m M_\nu^\dagger P_\mu M_\nu = P_\mu$. Moreover if P_μ is of rank-one, then it corresponds to a stationary state $\bar{\rho} = P_\mu$ of the Markov process (2.2), i.e., for any μ, $\frac{M_\mu\bar{\rho}M_\mu^\dagger}{\mathrm{Tr}\left(M_\mu\bar{\rho}M_\mu^\dagger\right)} = \bar{\rho}$.

2.6.2 Stability of discrete-time quantum filters

Consider the Markov chain (2.2). Assume that we do not know precisely the initial state ρ_0 and we have at our disposal an estimate ρ_0^e. Assume also that we know the measurement results μ_k. It is then natural to consider the following recursive update of our estimation ρ_{k+1}^e using the knowledge of μ_k and the previous estimate ρ_k^e :

$$\rho_{k+1}^e = \mathbb{M}_{\mu_k}(\rho_k^e). \tag{2.3}$$

Remark that the probability $\mathrm{Tr}\left(M_\mu \rho_k M_\mu^\dagger\right)$ to get $\mu_k = \mu$ depends on the hidden state ρ_k and not on ρ_k^e. The following theorem ensures the stability of such estimation process known as a discrete-time quantum filter.

Theorem 2.1 (Rouchon [82]). *Consider a Markov chain (ρ_k, ρ_k^e) satisfying Equations (2.2) and (2.3). Then $F(\rho_k, \rho_k^e) = \mathrm{Tr}^2\left(\sqrt{\sqrt{\rho_k}\rho_k^e\sqrt{\rho_k}}\right)$ is a submartingale, i.e.,*

$$\mathbb{E}\left[F(\rho_{k+1}, \rho_{k+1}^e)|\rho_k, \rho_k^e\right] \geq F(\rho_k, \rho_k^e).$$

2.7 Imperfect measurement

In this section, we consider repeated and imperfect measurements on a quantum system. In fact, by imperfect measurements, we mean both unread measurements performed by the environment (decoherence) and active measurements performed by non-ideal detectors. For more details on the content of this section, we refer to [90].

2.7.1 Measurement model

Consider a single ideal measurement. Take a set of Kraus operators M_q attached to the ideal detections (which are provided by the ideal sensors) for $q \in \{1, \ldots, m\}$. These Kraus operators satisfy $\sum_{q=1}^m M_q^\dagger M_q = \mathbb{1}$, for some $m \in \mathbb{N}$. We assume that the real sensors provide an outcome μ' that is a random variable in the set $\{1, \ldots, m'\}$, for some $m' \in \mathbb{N}$. Suppose that we know the correlation between the events $\mu = q$ and $\mu' = p$ which is given through the stochastic matrix $\eta \in \mathbb{R}^{m' \times m}$:

$$\eta_{p,q} = \mathbb{P}\left(\mu' = p|\mu = q\right). \tag{2.4}$$

Since $\eta_{p,q} \geq 0$ and for each q, $\sum_{p=1}^{m'} \eta_{p,q} = 1$, the matrix η is a left stochastic matrix.

Consider now a sequence of discrete-time POVM measurements. We denote by ρ_k the state of the system at time step k, with the initial state ρ_0 in $\mathcal{D}$. Assume $M_{q;k}$ denotes the Kraus operator corresponding to the k^{th} ideal measurement for $q \in \{1, \ldots, m\}$. We have

$$\mathbb{E}\left[\rho_{k+1}|\rho_k, \mu_k = q\right] = \mathbb{M}_{q;k}(\rho_k) := \frac{M_{q;k}\rho_k M_{q;k}^\dagger}{\mathrm{Tr}\left(M_{q;k}\rho_k M_{q;k}^\dagger\right)}, \tag{2.5}$$

with $\mathbb{P}(\mu_k = q|\rho_k) = \mathrm{Tr}\left(M_{q;k}\rho_k M_{q;k}^\dagger\right)$. Also let η^k be the stochastic matrix at step k. The optimal estimate is defined as

$$\widehat{\rho}_k = \mathbb{E}\left[\rho_k|\mathcal{A}_{k-1}\right], \tag{2.6}$$

where the event $\mathcal{A}_{k-1}$ is given by $\mathcal{A}_{k-1} := (\rho_0, \mu'_0 = p_0, \ldots, \mu'_{k-1} = p_{k-1})$.

The following theorem provides a recursive equation for $\widehat{\rho}_k$.

Theorem 2.2 (Somaraju et al. [90]). *The optimal estimate $\widehat{\rho}_k$ satisfies the following recursive equation*

$$\widehat{\rho}_{k+1} = \mathbb{L}_{p_k}(\widehat{\rho}_k) := \frac{\sum_{q=1}^m \eta^k_{p_k,q} M_{q;k}\widehat{\rho}_k M_{q;k}^\dagger}{\mathrm{Tr}\left(\sum_{q=1}^m \eta^k_{p_k,q} M_{q;k}\widehat{\rho}_k M_{q;k}^\dagger\right)}, \tag{2.7}$$

if $\mu'_k = p_k$ (where p_k is a random variable taking values in the set $\{1,\ldots,m'\}$). Moreover, we have

$$\mathbb{P}\left(\mu'_k = p_k|\mathcal{A}_{k-1}\right) = \mathrm{Tr}\left(\sum_{q=1}^m \eta^k_{p_k,q} M_{q;k}\widehat{\rho}_k M_{q;k}^\dagger\right). \tag{2.8}$$

In the rest of this section, we provide an alternative proof of the above theorem which simplifies the original arguments of [90].

Proof. We start by proving Equation (2.8). We have

$$\mathrm{Tr}\left(\sum_{q=1}^m \eta^k_{p_k,q} M_{q;k}\widehat{\rho}_k M_{q;k}^\dagger\right) = \mathrm{Tr}\left(\sum_{q=1}^m \mathbb{P}\left(\mu'_k = p_k|\mu_k = q\right) M_{q;k}\mathbb{E}\left[\rho_k|\mathcal{A}_{k-1}\right] M_{q;k}^\dagger\right),$$

where we have replaced the expressions of $\eta^k_{p_k,q}$ and $\widehat{\rho}_k$ given respectively in (2.4) and (2.6). Using the fact that we can commute the expectation and the linear function trace, the right hand side term of the above equation becomes equal to

$$\begin{aligned}
\mathbb{E}\left[\mathrm{Tr}\left(\sum_{q=1}^m \mathbb{P}\left(\mu'_k = p_k|\mu_k = q\right) M_{q;k}\rho_k M_{q;k}^\dagger\right)|\mathcal{A}_{k-1}\right] &= \\
\mathbb{E}\left[\sum_{q=1}^m \mathbb{P}\left(\mu'_k = p_k|\mu_k = q\right)\mathrm{Tr}\left(M_{q;k}\rho_k M_{q;k}^\dagger\right)|\mathcal{A}_{k-1}\right] &= \\
\mathbb{E}\left[\sum_{q=1}^m \mathbb{P}\left(\mu'_k = p_k|\mu_k = q\right)\mathbb{P}\left(\mu_k = q|\rho_k\right)|\mathcal{A}_{k-1}\right] &= \\
\mathbb{E}\left[\sum_{q=1}^m \mathbb{P}\left(\mu'_k = p_k|\mu_k = q, \rho_k\right)\mathbb{P}\left(\mu_k = q|\rho_k\right)|\mathcal{A}_{k-1}\right]&,
\end{aligned} \tag{2.9}$$

where in the second equality, we use that $\mathbb{P}(\mu_k = q|\rho_k) = \mathrm{Tr}\left(M_{q;k}\rho_k M_{q;k}^\dagger\right)$ and in the last equality, we use $\mathbb{P}(\mu_k' = p_k|\mu_k = q) = \mathbb{P}(\mu_k' = p_k|\mu_k = q, \rho_k)$, since by relation (2.4), the probability of $\mu_k' = p_k$ is characterized only by knowing the value of $\mu_k = q$.

The term (2.9) is equal to

$$\mathbb{E}\left[\mathbb{P}(\mu_k' = p_k|\rho_k)|\mathcal{A}_{k-1}\right] = \mathbb{E}\left[\mathbb{E}\left[I_{\mu_k'=p_k}|\rho_k\right]|\mathcal{A}_{k-1}\right] = \mathbb{E}\left[I_{\mu_k'=p_k}|\mathcal{A}_{k-1}\right] = \mathbb{P}(\mu_k' = p_k|\mathcal{A}_{k-1}),$$

where $I_{\mu_k'=p_k}$ denotes the indicator function, i.e., it is equal to one if $\mu_k' = p_k$ and is equal to zero if $\mu_k' \neq p_k$. The above formula is obtained using the facts that $\mathbb{E}[\mathbb{E}[A|B]] = \mathbb{E}[A]$ and $\mathbb{E}[I_A] = \mathbb{P}(A)$. This finishes the proof of the relation (2.8).

In order to prove the recursive relation (2.7), we apply the formula (2.8). We have to show the following equation

$$\widehat{\rho}_{k+1} = \frac{\sum_{q=1}^m \eta_{p_k,q}^k M_{q;k}\widehat{\rho}_k M_{q;k}^\dagger}{\mathbb{P}(\mu_k' = p_k|\mathcal{A}_{k-1})}.$$

This is equivalent to proving that

$$\mathbb{E}[\rho_{k+1}|\mathcal{A}_k]\,\mathbb{P}(\mu_k' = p_k|\mathcal{A}_{k-1}) = \sum_{q=1}^m \eta_{p_k,q}^k M_{q;k}\widehat{\rho}_k M_{q;k}^\dagger = \sum_{q=1}^m \mathbb{P}(\mu_k' = p_k|\mu_k = q)\, M_{q;k}\widehat{\rho}_k M_{q;k}^\dagger. \tag{2.10}$$

By Equation (2.5), we get

$$M_{q;k}\rho_k M_{q;k}^\dagger = \mathbb{E}[\rho_{k+1}|\rho_k, \mu_k = q]\,\mathrm{Tr}\left(M_{q;k}\rho_k M_{q;k}^\dagger\right) = \mathbb{E}[\rho_{k+1}|\rho_k, \mu_k = q]\,\mathbb{P}(\mu_k = q|\rho_k).$$

Now we compute the expectation with respect to ρ_k conditioned on the event $\mathcal{A}_{k-1}$. We find the following expression:

$$\mathbb{E}\left[M_{q;k}\rho_k M_{q;k}^\dagger|\mathcal{A}_{k-1}\right] = M_{q;k}\widehat{\rho}_k M_{q;k}^\dagger = \mathbb{E}\left[\mathbb{E}[\rho_{k+1}|\rho_k, \mu_k = q]\,\mathbb{P}(\mu_k = q|\rho_k)|\mathcal{A}_{k-1}\right].$$

Therefore, the right hand side of (2.10) can be written as

$$\begin{aligned}
&\sum_{q=1}^m \mathbb{P}(\mu_k' = p_k|\mu_k = q)\, M_{q;k}\widehat{\rho}_k M_{q;k}^\dagger = \\
&\sum_{q=1}^m \mathbb{P}(\mu_k' = p_k|\mu_k = q)\,\mathbb{E}\left[\mathbb{E}[\rho_{k+1}|\rho_k, \mu_k = q]\,\mathbb{P}(\mu_k = q|\rho_k)|\mathcal{A}_{k-1}\right] = \\
&\sum_{q=1}^m \mathbb{P}(\mu_k' = p_k|\mu_k = q)\,\mathbb{E}\left[\frac{\mathbb{E}[\rho_{k+1}I_{\mu_k=q}|\rho_k]}{\mathbb{P}(\mu_k = q|\rho_k)}\mathbb{P}(\mu_k = q|\rho_k)|\mathcal{A}_{k-1}\right] = \\
&\sum_{q=1}^m \mathbb{P}(\mu_k' = p_k|\mu_k = q)\,\mathbb{E}\left[\rho_{k+1}I_{\mu_k=q}|\mathcal{A}_{k-1}\right],
\end{aligned} \tag{2.11}$$

where we have used the fact that $\mathbb{E}\left[A|B,C\right]=\frac{\mathbb{E}\left[AI_C|B\right]}{\mathbb{P}(C|B)}$.

Since

$$\mathbb{E}\left[\rho_{k+1}I_{\mu_k'=p_k}|\mathcal{A}_{k-1}\right]= \\ \mathbb{E}\left[\rho_{k+1}|\mu_k'=p_k,\mathcal{A}_{k-1}\right]\mathbb{P}\left(\mu_k'=p_k|\mathcal{A}_{k-1}\right)=\mathbb{E}\left[\rho_{k+1}|\mathcal{A}_k\right]\mathbb{P}\left(\mu_k'=p_k|\mathcal{A}_{k-1}\right),$$

in order to show (2.10), it will be enough to show that the right term in Equation (2.11) is equal to $\mathbb{E}\left[\rho_{k+1}I_{\mu_k'=p_k}|\mathcal{A}_{k-1}\right]$, i.e.,

$$\sum_{q=1}^{m}\mathbb{P}\left(\mu_k'=p_k|\mu_k=q\right)\mathbb{E}\left[\rho_{k+1}I_{\mu_k=q}|\mathcal{A}_{k-1}\right]=\mathbb{E}\left[\rho_{k+1}I_{\mu_k'=p_k}|\mathcal{A}_{k-1}\right], \tag{2.12}$$

Equation (2.12) is equivalent to

$$\sum_{q=1}^{m}\mathbb{P}\left(\mu_k'=p_k|\mu_k=q,\mathcal{A}_{k-1}\right)\mathbb{E}\left[\rho_{k+1}I_{\mu_k=q}|\mathcal{A}_{k-1}\right]=\mathbb{E}\left[\rho_{k+1}I_{\mu_k'=p_k}|\mathcal{A}_{k-1}\right], \tag{2.13}$$

where we have used the fact that $\mathbb{P}\left(\mu_k'=p_k|\mu_k=q,\mathcal{A}_{k-1}\right)=\mathbb{P}\left(\mu_k'=p_k|\mu_k=q\right)$, since by relation (2.4), the probability of the event $\mu_k'=p_k$ is determined only by knowing $\mu_k=q$.

The left hand side of the above formulas is equal to

$$\sum_{q=1}^{m}\mathbb{P}\left(\mu_k'=p_k|\mu_k=q,\mathcal{A}_{k-1}\right)\mathbb{E}\left[\rho_{k+1}I_{\mu_k=q}|\mathcal{A}_{k-1}\right]= \\ \sum_{q=1}^{m}\mathbb{P}\left(\mu_k'=p_k|\mu_k=q,\mathcal{A}_{k-1}\right)\mathbb{E}\left[\rho_{k+1}|\mu_k=q,\mathcal{A}_{k-1}\right]\mathbb{P}\left(\mu_k=q|\mathcal{A}_{k-1}\right)= \\ \sum_{q=1}^{m}\mathbb{P}\left(\mu_k'=p_k,\mu_k=q|\mathcal{A}_{k-1}\right)\mathbb{E}\left[\rho_{k+1}|\mu_k=q,\mathcal{A}_{k-1}\right]= \\ \sum_{q=1}^{m}\mathbb{P}\left(\mu_k=q|\mu_k'=p_k,\mathcal{A}_{k-1}\right)\mathbb{P}\left(\mu_k'=p_k|\mathcal{A}_{k-1}\right)\mathbb{E}\left[\rho_{k+1}|\mu_k=q,\mathcal{A}_{k-1}\right].$$

Here for the first equality, we have used

$$\mathbb{E}\left[\rho_{k+1}I_{\mu_k=q}|\mathcal{A}_{k-1}\right]=\mathbb{E}\left[\rho_{k+1}|\mu_k=q,\mathcal{A}_{k-1}\right]\mathbb{P}\left(\mu_k=q|\mathcal{A}_{k-1}\right).$$

Now, in the same way $\mathbb{E}\left[\rho_{k+1}I_{\mu_k'=p_k}|\mathcal{A}_{k-1}\right]=\mathbb{E}\left[\rho_{k+1}|\mu_k'=p_k,\mathcal{A}_{k-1}\right]\mathbb{P}\left(\mu_k'=p_k|\mathcal{A}_{k-1}\right)$, and therefore demonstrating (2.13) is equivalent to showing the following relation

$$\sum_{q=1}^{m}\mathbb{P}\left(\mu_k=q|\mu_k'=p_k,\mathcal{A}_{k-1}\right)\mathbb{P}\left(\mu_k'=p_k|\mathcal{A}_{k-1}\right)\mathbb{E}\left[\rho_{k+1}|\mu_k=q,\mathcal{A}_{k-1}\right]= \\ \mathbb{E}\left[\rho_{k+1}|\mu_k'=p_k,\mathcal{A}_{k-1}\right]\mathbb{P}\left(\mu_k'=p_k|\mathcal{A}_{k-1}\right).$$

Finally, we have to show

$$\sum_{q=1}^{m} \mathbb{P}\left(\mu_k = q | \mu'_k = p_k, \mathcal{A}_{k-1}\right) \mathbb{E}\left[\rho_{k+1} | \mu_k = q, \mathcal{A}_{k-1}\right] = \mathbb{E}\left[\rho_{k+1} | \mu'_k = p_k, \mathcal{A}_{k-1}\right],$$

which is equivalent to

$$\sum_{q=1}^{m} \mathbb{P}\left(\mu_k = q | \mu'_k = p_k, \mathcal{A}_{k-1}\right) \mathbb{E}\left[\rho_{k+1} | \mu_k = q, \mu'_k = p_k, \mathcal{A}_{k-1}\right] = \mathbb{E}\left[\rho_{k+1} | \mu'_k = p_k, \mathcal{A}_{k-1}\right], \tag{2.14}$$

where we have used $\mathbb{E}\left[\rho_{k+1} | \mu_k = q, \mathcal{A}_{k-1}\right] = \mathbb{E}\left[\rho_{k+1} | \mu_k = q, \mu'_k = p_k, \mathcal{A}_{k-1}\right]$, since by Equation (2.5), knowing the value of μ'_k does not add further information to calculate the expectation $\mathbb{E}\left[\rho_{k+1} | \mu_k = q\right]$.

But Equation (2.14) results from the following elementary fact

$$\mathbb{E}\left[X | \mathcal{B}\right] = \sum_{y} \mathbb{E}\left[X | Y = y, \mathcal{B}\right] \mathbb{P}\left(Y = y | \mathcal{B}\right).$$

In the above equation, set $X = \rho_{k+1}$, $Y = \mu_k$, $q = y$, and $\mathcal{B}$ the event $\mu'_k = p_k$, we then necessarily have Equation (2.14).

This finishes the proof of the theorem. □

2.7.2 Stability with respect to initial condition

Consider the recursive equation (2.7). Assume that we do not have access to the initial state ρ_0. We cannot compute the optimal estimate $\widehat{\rho}_k$. The natural estimation can be still given by the recursive formula (2.7) based on the real measurement outcomes $\mu'_0, \ldots, \mu'_{k-1}$ and an arbitrary initial estimate state $\widehat{\rho}^e_0$. Thus if $\mu'_k = p$, we define $\widehat{\rho}^e_k$ for $k \geq 1$ as follows

$$\widehat{\rho}^e_{k+1} = \mathbb{L}_p(\widehat{\rho}^e_k). \tag{2.15}$$

The recursive formula (2.15) is valid as soon as $\sum_q \operatorname{Tr}\left(\eta^k_{p,q} M_{q;k} \widehat{\rho}^e_k M^\dagger_{q;k}\right) > 0$, which is automatically satisfied when $\widehat{\rho}^e_k$ is of full rank. (When $\sum_q \operatorname{Tr}\left(\eta^k_{p,q} M_{q;k} \widehat{\rho}^e_k M^\dagger_{q;k}\right) = 0$, we can still define the value of $\widehat{\rho}^e_{k+1}$, see more details in [90].) We state now a theorem ensuring the stability of such estimation procedure whatever the initial state $\widehat{\rho}^e_0$ is.

Theorem 2.3 (Somaraju et al. [90]). *Suppose $\widehat{\rho}^e_k$ satisfies the recursive relation (2.15) with an arbitrary initial density operator $\widehat{\rho}^e_0$. Then the fidelity between $\widehat{\rho}_k$ and $\widehat{\rho}^e_k$ defined by*

$$F(\widehat{\rho}_k, \widehat{\rho}^e_k) = \operatorname{Tr}^2\left(\sqrt{\sqrt{\widehat{\rho}_k}\,\widehat{\rho}^e_k\sqrt{\widehat{\rho}_k}}\right)$$

is a submartingale in the following sense:

$$\mathbb{E}\left[F(\widehat{\rho}_{k+1}, \widehat{\rho}^e_{k+1}) | \widehat{\rho}_0, \ldots, \widehat{\rho}_k, \widehat{\rho}^e_0, \ldots, \widehat{\rho}^e_k\right] \geq F(\widehat{\rho}_k, \widehat{\rho}^e_k).$$

With this preliminary introduction to models of discrete-time open quantum systems, we can study the remaining chapters.

Chapter 3

Fidelity based stabilization of the photon box

Contents

Dans le but de réaliser un traitement robuste de l'information quantique, l'une des tâches principale consiste à préparer et à protéger les différents états quantiques. Durant les quinze dernières années, l'application des paradigmes de rétroaction basée sure la mesure ont été étudiés par des nombreux physiciens [107, 97, 34, 103, 71, 47] comme une solution possible pour cette préparation robuste. Cependant, la plupart de ces efforts sont restés à un niveau théorique et n'ont pas été en mesure de donner lieu à des expériences abouties. Cela est essentiellement dû à la nécessité de simuler, en parallèle avec le système, un filtre quantique [16] qui fournit une estimation de l'état du système basée sur l'historique des sauts quantiques induit par le processus de mesure. En effet, il est en général difficile d'effectuer de telles simulations en temps réel. Dans ce chapitre, nous considérons que la loi de rétroaction proposée dans [36] dans le cadre de la boite à photons, un systéme quantique ouvert en temps discret, où nous avons pu effectuer ces calculs en temps réel (voir [44] pour une description détaillée de ce système électrodynamique quantique en cavité).

En tenant compte du postulat que la mesure quantique induit une projection, les protocoles de mesure les plus pratiques dans le but de réaliser une rétroaction sont les mesures quantique non-destructive (QND) [24, 96, 99]. Comme nous l'avons vu au Chapitre 2, ce sont des mesures qui préservent la valeur de l'observable mesurée. En effet, en considérant un processus de mesure QND bien conçu où l'état quantique à préparer est un état propre de l'opérateur de mesure, le processus de mesure n'est pas un obstacle pour la préparation de l'état, mais il peut même l'aider en ajoutant de la contrôlabilité ou de la stabilité.
Très souvent, un protocole de mesure QND bien choisi peut lui-même être considéré comme un outil de préparation pour les différents états quantiques. Noter que la conception de ces mesures n'est pas toujours pratique (voir par exemple [25]). Cependant, cette préparation est généralement non-déterministe et on ne peut pas s'assurer qu'elle converge vers l'état souhaité, sauf en répétant l'expérience plusieurs fois. La rétroaction peut être appliquée pour rendre ce processus déterministe [93, 71, 36].

Dans [32, 43, 40], des mesures QND sont exploités pour détecter et/ou produire des états fortement non-classiques de la lumière piégée dans une cavité supraconductrice (voir [44, Chapitre 5] pour une description de tels systèmes CQED et [27] pour les détails sur les modèles physiques utilisés pour des mesures QND de la lumière en utilisant des atomes). Pour un tel montage expérimental, nous détaillons et analysons ici un schéma de rétroaction qui stabilise le champ de la cavité vers n'importe quel état-nombre de photon (état de Fock). Ces états sont fortement non-classiques car leurs nombres de photons sont parfaitement définis. L'entrée de commande est de type classique et correspond è une impulsion cohérente de la lumière injectée à l'intérieur de la cavité entre les passages des atomes. La structure globale de ce schéma de rétroaction proposé est inspirée de [39] en utilisant une adaptation quantique de la structure observateur/contrôleur largement mise en œvre pour les systèmes classiques (voir par exemple [50, Chapitre 4]). Comme les mesures induisant des sauts quantiques et l'injection du champ de contrôle arrivent en temps discret, la partie d'observation du schéma de rétroaction proposée consiste en un filtre quantique à temps discret. En effet, le fait que la mesure arrive à temps discret nous fournit un premier prototype des systèmes quantiques où nous avons suffisamment du temps pour effectuer le filtrage quantique et calculer la loi de rétroaction basée sur la mesure à appliquer pour le contrôle.

Du point de vue de la modélisation mathématique, le filtre quantique évolue comme une chaîne de Markov à temps discret. L'état estimé est utilisé dans une rétroaction d'état, basée sur le modèle de Lyapunov. En considérant un candidat naturel pour la fonction de Lyapunov, la loi de rétroaction assure la diminution de son espérance sur le processus de Markov. Par conséquent, la valeur de la fonction de Lyapunov considérée au cours de la chaîne de Markov définit une sur-martingale. L'analyse de la convergence du système en boucle fermée est donc basée sur des outils plutôt classiques de l'analyse de stabilité stochastique [55]. Le contrôle de chaînes de Markov similaires a déjà été pris en compte. Dans [47], la rétroaction d'état est dérivée du contrôle optimal sensible au risque. En principe, nous pouvons appliquer ces techniques au cas étudié. Néanmoins, comme le nombre d'entréées scalaires de l'état est important (autour de 50 pour une matrice 10×10), le calcul de cette loi de rétroaction optimale n'est pas possible en temps réel. Dans [18], une stratégie de rétroaction de sortie statique est proposée afin de stabiliser un état pur. Malheureusement, nous ne pouvons pas appliquer cette méthode ici pour les deux raisons suivantes. Tout d'abord, nous avons seulement à notre disposition l'ensemble des opérateurs de déplacements cohérents comme les transformations unitaires sur l'état quantique de la cavité (oscillateur harmonique quantifié). Avec cet ensemble restreint, il n'est pas possible de stabiliser un état de Fock en prenant l'amplitude de déplacement comme une fonction statique du résultat de la mesure. Deuxiement, il semble que l'adaptation d'une telle rétroaction de sortie statique au retard de la mesure est une question ouverte.

L'une des caractéristiques particulières du systèmes pris en compte dans ce chapitre correspond à un retard non-négligeable dans le processus de rétroaction. Comme expliqué au Chapitre 1, dans le dispositif expérimental considéré dans ce chapitre, il existe un délai de τ étapes entre le processus de mesure et l'injection de rétroaction. Dans ce chapitre, nous proposons une adaptation du filtre quantique, basée sur une version stochastique d'un prédicteur de type Kalman, qui prend en compte ce retard par la prédiction de l'état actuel du système sans avoir l'accès au résultat de la dernière détection.

Dans la Section 3.1, nous décrivons brièvement le modèle physique de la boite à photons et de sa dynamique de saut. La Section 3.2 est consacrée au comportement en boucle ouverte : nous montrons dans le Théorème 3.1 que le processus de mesure QND, sans aucune injection de contrôle supplémentaire, permet une préparation non-déterministe des états de Fock. En effet, nous constatons que la chaîne de Markov associée converge nécessairement vers un état de Fock et la probabilité de convergence vers un état de Fock fixé est donnée par sa population sur l'état initial. En outre, dans la Proposition 3.1, nous montrons que le système linearisé autour d'un état de Fock fixé en boucle ouverte admet un exposant de Lyapunov strictement négatif (voir la Section A.3 de l'Annexe A pour une définition de l'exposant de Lyapunov). Dans la Section 3.3, nous proposons un modèle de Lyapunov basé sur la rétroaction permettant de stabiliser globalement le système avec le retard en boucle fermée autour d'un état de Fock désiré. Le Théorème 3.2 prouve la convergence presque sure des trajectoires du système en boucle fermée vers l'état de Fock cible. En outre, grâce à la Proposition 3.2, nous prouvons que le système linearisé autour de l'état de Fock cible en boucle fermée admet un exposant de Lyapunov strictement négatif. Enfin, dans la Section 3.4, nous proposons une brève discussion sur le filtre quantique considéré, et en prouvant un principe de séparation plutôt général (Théorème 3.3), nous montrons une robustesse semi-globale par rapport à la connaissance de l'état initial du système. Aussi, à travers une brève analyse du système-observateur linearisé autour de l'état de Fock cible et en appliquant les Propositions 3.1 et 3.2, nous montrons que son plus grand exposant de Lyapunov est également strictement négatif (Proposition 3.3).

Une version préliminaire, en l'absence de tout retard, a été pris en compte dans [68]. Le schéma de compensation du retard a été introduit dans [36]. Les résultats de ce chapitre sont basés sur [3].

In the aim of achieving a robust processing of quantum information, one of the main tasks is to prepare and to protect various quantum states. Through the last 15 years, the application of measurement-based feedback paradigms has been investigated by many physicists [107, 97, 34, 103, 71, 47] as a possible solution for this robust preparation. However, most of these efforts have remained at a theoretical level and have not been able to give rise to successful experiments. This is essentially due to the necessity of simulating, in parallel to the system, a quantum filter [16] providing an estimate of the state of the system based on the historic of quantum jumps induced by the measurement process. Indeed, it is in general difficult to perform such simulations in real-time. In this chapter, we consider the feedback law proposed in [36] for the photon box, a discrete-time open quantum system, where we actually have the time to perform these computations in real-time (see [44] for a detailed description of this cavity quantum electrodynamics system).

Taking into account the measurement-induced quantum projection postulate, the most practical measurement protocols in the aim of feedback control are the quantum non-demolition (QND) measurements [24, 96, 99]. As we have seen in Chapter 2, these are the measurements which preserve the value of the measured observable. Indeed, by considering a well-designed QND measurement process where the quantum state to be prepared is an eigenstate of the measurement operator, the measurement process is not an obstacle for the state preparation but can even help by adding some controllability and/or stability.
In fact, very often a well-chosen QND measurement protocol can itself be considered as a preparation tool for various quantum states. Note that the design of such measurements is not always practical (see e.g., [25]). However, this preparation is generally non-deterministic and one cannot make sure that it converges towards the desired state except by repeating the experiment many times. The feedback can be applied to make this process deterministic [93, 71, 36].

In [32, 43, 40] QND measures are exploited to detect and/or produce highly non-classical states of light trapped in a super-conducting cavity (see [44, Chapter 5] for a description of such CQED systems and [27] for detailed physical models with QND measures of light using atoms). For such experimental setup, we detail and analyze here a feedback scheme that stabilizes the cavity field towards any photon-number states (Fock states). Such states are strongly non-classical since their photon numbers are perfectly defined. The control input is of classical type and corresponds to a coherent light-pulse injected inside the cavity between atom passages. The overall structure of the proposed feedback scheme is inspired by [39] using a quantum adaptation of the observer/controller structure widely used for classical systems (see e.g., [50, Chapter 4]). As the measurement-induced quantum jumps and the controlled field injection happen in a discrete-time manner, the observer part of the proposed feedback scheme consists in a discrete-time quantum filter. Indeed, the discreteness of the measurement process provides us a first prototype of quantum systems, where we actually have enough time to perform the quantum filtering and to compute the measurement-based feedback law to be applied as the controller.

From a mathematical modeling point of view, the quantum filter evolves through a discrete-time Markov chain. The estimated state is used in a state feedback, based on a Lyapunov design. By considering a natural candidate for the Lyapunov function, the feedback law ensures the decrease of its expectation over the Markov process. Therefore, the value of the considered Lyapunov function over the Markov chain defines a supermartingale. The convergence analysis of the closed-loop system is therefore based on some rather classical tools from stochastic stability analysis [55]. The control of similar discrete-time quantum Markov chains has been already considered. In [47] the state feedback is derived from risk-sensitive optimal control. In principle, one can apply these techniques to our case. Nevertheless, since the number of scalar entries of the state is large (around 50 for a 10×10 density matrix), computations of this optimal feedback law is not possible in real-time. In [18] a static output feedback strategy is proposed to stabilize a pure state. Unfortunately, we cannot apply this method here for the following two reasons. Firstly, we have only at our disposal the set of coherent displacement operators as unitary transformations on the cavity quantum state (quantized harmonic oscillator). With this restricted set, it is not possible to stabilize a Fock state by taking the displacement amplitude as a static function of measurement outcome. Secondly, it appears that adaptation of such static output feedback to measurement delay is an open question.

One of the particular features of the system considered in this chapter corresponds to a non-negligible delay in the feedback process. As explained in Chapter 1, in the experimental setup considered through this chapter, there exists a delay of τ steps between the measurement process and the feedback injection. Through this chapter, we propose an adaptation of the quantum filter, based on a stochastic version of a Kalman-type predictor, which takes into account this delay by predicting the actual state of the system without having access to the result of τ last detections.

In Section 3.1, we briefly describe the physical model of the photon box and its jump dynamics. Section 3.2 is devoted to the open-loop behavior: we show in Theorem 3.1 that the QND measurement process, without any additional controlled injection, allows a non-deterministic preparation of the Fock states. Indeed, we will see that the associated Markov chain converges necessarily towards a Fock state and that the probability of the convergence towards a fixed Fock state is given by its population over the initial state. Also, in Proposition 3.1, we will show that the linearized open-loop system around a fixed Fock state admits strictly negative Lyapunov exponents (see Section A.3 in Appendix A for a definition of the Lyapunov exponent). In Section 3.3, we propose a Lyapunov-based feedback design allowing to stabilize globally the delayed closed-loop system around a desired Fock state. Theorem 3.2 proves the almost sure convergence of the trajectories of the closed-loop system towards the target Fock state. Also, through Proposition 3.2, we will prove that the linearized closed-loop system around the target Fock state admits strictly negative Lyapunov exponents. Finally in Section 3.4, we propose a brief discussion on the considered quantum filter, and by proving a rather general separation principle (Theorem 3.3), we show a semi-global robustness with respect to the knowledge of the initial state of the system. Also, through a brief analysis of the linearized system-observer around the target Fock state and by applying Propositions 3.1 and 3.2, we show that its largest Lyapunov exponent is also strictly negative (Proposition 3.3).

A preliminary version, in the absence of any delay was considered in [68]. The delay compensation scheme was introduced in [36]. The results of this chapter are based on [3].

3.1 Physical model of the photon box and its jump dynamics

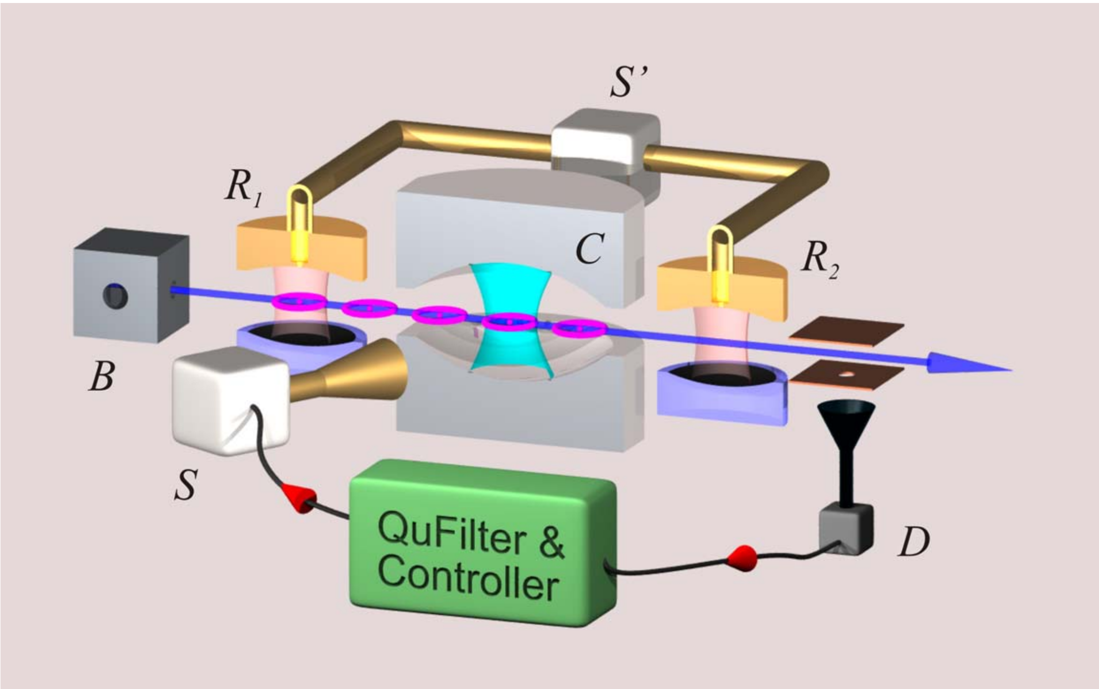

Figure 3.1: The ENS photon box; atoms get out of box B one by one, undergo then a first Rabi pulse in Ramsey zone R_1, become entangled with electromagnetic field trapped in C, undergo a second Rabi pulse in Ramsey zone R_2, and finally are measured in the detector D. The control corresponds to a coherent displacement of amplitude $\alpha \in \mathbb{C}$ that is applied via the microwave source S between two atom passages.

As illustrated by Figure 3.1, the system consists in C a high-Q microwave cavity, in B a box producing Rydberg atoms, in R_1 and R_2 two low-Q Ramsey cavities and in D an atom detector. The dynamics model is discrete in time and takes into account the measurement back-action. Each time step is indexed by the integer k corresponding to atom number k coming from B, submitted then to a first Ramsey $\pi/2$-pulse in R_1, crossing the cavity C and being entangled with it, submitted to a second $\pi/2$-pulse in R_2 and finally being measured in D. The state of the cavity is associated to a quantized mode. The control corresponds to a coherent displacement of amplitude $\alpha \in \mathbb{C}$ that is applied via the microwave source S between two atom passages. We consider a finite-dimensional approximation of this quantized mode and take a truncation to $n^{\max}$ photons. The cavity state is thus approximated by the Hilbert space $\mathbb{C}^{n^{\max}+1}$. It admits $(|0\rangle, |1\rangle, \ldots, |n^{\max}\rangle)$ as an orthonormal basis. Each basis vector $|n\rangle \in \mathbb{C}^{n^{\max}+1}$ corresponds to a pure state, called Fock state, where the cavity has exactly n photons with $n \in \{0, \ldots, n^{\max}\}$.

In this Fock states basis the number operator $\mathbf{N}$ corresponds to the diagonal matrix

$$\mathbf{N} = \operatorname{diag}(0, 1, \ldots, n^{\max}).$$

The annihilation operator truncated to $n^{\max}$ photons is denoted by $\mathbf{a}$. It corresponds to the upper 1-diagonal matrix filled with $(\sqrt{1}, \ldots, \sqrt{n^{\max}})$:

$$\mathbf{a}\,|0\rangle = 0, \quad \mathbf{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle \text{ for } n = 1, \ldots, n^{\max}.$$

The truncated creation operator denoted by $\mathbf{a}^\dagger$ is the Hermitian conjugate of $\mathbf{a}$. Notice that we still have $\mathbf{N} = \mathbf{a}^\dagger\mathbf{a}$, but truncation does not preserve the usual commutation $[\mathbf{a}, \mathbf{a}^\dagger] = 1$ that is only valid when $n^{\max} = +\infty$.

The delay attached to the atoms flying between cavity and detector corresponds to a measurement delay on system output. On system input we have no delay. It is well-known that systems with constant output delay and without input delay can be described also as systems with constant input delay and without output delay. We consider here this second description where the delay is attached to the input: the control elaborated at step k, α_k, is subject to a delay of τ steps, τ representing the number of flying atoms between the cavity C and the detector D. Just after the measurement of the atom number $k-1$, the state of the cavity is described by the density matrix ρ_k belonging to the following set of well-defined density matrices:

$$\mathcal{X} = \left\{\rho \in \mathbb{C}^{(n^{\max}+1)^2} \quad | \quad \rho = \rho^\dagger,\ \operatorname{Tr}(\rho) = 1,\ \rho \geq 0\right\}. \tag{3.1}$$

The random evolution of this state ρ_k can be modeled through a discrete-time Markov process that will be described below (see [36, 27] and the references therein explaining the physical modeling assumptions). Let us denote by $\alpha_k \in \mathbb{C}$ the control at step k. Then ρ_{k+1}, the cavity state after measurement of atom k, is given by

$$\rho_{k+1} = \mathbb{M}_{\mu_k}(\rho_{k+\frac{1}{2}}), \quad \rho_{k+\frac{1}{2}} = \mathbb{D}_{\alpha_{k-\tau}}(\rho_k), \tag{3.2}$$

where

- $\mu_k \in \{g, e\}$, $\mathbb{M}_g(\rho) = \frac{M_g \rho M_g^\dagger}{\operatorname{Tr}\left(M_g \rho M_g^\dagger\right)}$, $\mathbb{M}_e(\rho) = \frac{M_e \rho M_e^\dagger}{\operatorname{Tr}\left(M_e \rho M_e^\dagger\right)}$ with operators $M_g = \cos(\varphi_0 \mathbb{1} + \vartheta\mathbf{N})$ and $M_e = \sin(\varphi_0 \mathbb{1} + \vartheta\mathbf{N})$ (φ_0, ϑ constant parameters). For any $n \in \{0, \ldots, n^{\max}\}$, we set $\varphi_n = \varphi_0 + n\vartheta$;

- $\mathbb{D}_\alpha(\rho) = D_\alpha \rho D_\alpha^\dagger$, where the unitary displacement operator D_α is given by $D_\alpha = e^{\alpha\mathbf{a}^\dagger - \alpha^*\mathbf{a}}$. In open-loop, $\alpha = 0$, $D_0 = \mathbb{1}$ (identity operator) and $\mathbb{D}_0(\rho) = \rho$. Note that $D_\alpha^\dagger = D_{-\alpha}$;

- μ_k is a random variable taking the value g when the atom k is detected in g (resp. e when the atom k is detected in e) with probability

$$P_{g,k} = \mathrm{Tr}\left(M_g \rho_{k+\frac{1}{2}} M_g^\dagger\right) \quad \left(\text{resp. } P_{e,k} = \mathrm{Tr}\left(M_e \rho_{k+\frac{1}{2}} M_e^\dagger\right)\right); \tag{3.3}$$

- the control elaborated at step k, α_k, is subject to a delay of τ steps, τ being the number of flying atoms between the cavity C and the detector D.

We will assume throughout this chapter that the parameters φ_0, ϑ are chosen in order to have M_g, M_e invertible and such that the spectrum of $M_g^\dagger M_g = M_g^2$ and $M_e^\dagger M_e = M_e^2$ are non-degenerate. This implies that the nonlinear operators $\mathbb{M}_g$ and $\mathbb{M}_e$ are well-defined for all $\rho \in \mathcal{X}$, and that $\mathbb{M}_g(\rho)$ and $\mathbb{M}_e(\rho)$ belongs also to the state space $\mathcal{X}$ defined by (3.1). We note that M_g and M_e commute, are diagonal in the Fock basis and satisfy $M_g^\dagger M_g + M_e^\dagger M_e = \mathbb{1}$. This means that the measurement process achieves a quantum non-demolition (QND) measurement for the physical observables given by projection operators over the states $\{\,|n\rangle \in \mathbb{C}^{n^{\max}+1} \mid \; n \in \{0, \ldots, n^{\max}\}\}$. The Kraus map associated to this Markov process is given by

$$\mathbb{K}_\alpha(\rho) = M_g D_\alpha \rho D_\alpha^\dagger M_g^\dagger + M_e D_\alpha \rho D_\alpha^\dagger M_e^\dagger. \tag{3.4}$$

It corresponds to the expectation value of ρ_{k+1} knowing ρ_k and $\alpha_{k-\tau}$:

$$\mathbb{E}\left[\rho_{k+1} \mid \rho_k, \alpha_{k-\tau}\right] = \mathbb{K}_{\alpha_{k-\tau}}(\rho_k). \tag{3.5}$$

3.2 Open-loop behavior

We consider in this section the following dynamics

$$\rho_{k+1} = \mathbb{M}_{\mu_k}(\rho_k), \tag{3.6}$$

obtained from (3.2) when $\alpha_{k-\tau} \equiv 0$.

3.2.1 Simulations

Figure 3.2 corresponds to 100 realizations of this Markov process with $n^{\max} = 10$ photons, $\vartheta = \frac{2}{10}$ and $\varphi_0 = \frac{\pi}{4} - 3\vartheta$. For each realization, ρ_0 is initialized to the same coherent state $\mathbb{D}_{\sqrt{3}}(|0\rangle\langle 0|)$ with $\mathrm{Tr}(\mathbf{N}\rho_0) \approx 3$ as mean photon number. We observe that $\langle 3|\rho_k|3\rangle$ tends either to 1 or 0. Since the ensemble average curve is almost constant, the proportion of trajectories for which $\langle 3|\rho_k|3\rangle$ tends to 1 is given approximatively by $\langle 3|\rho_0|3\rangle$.

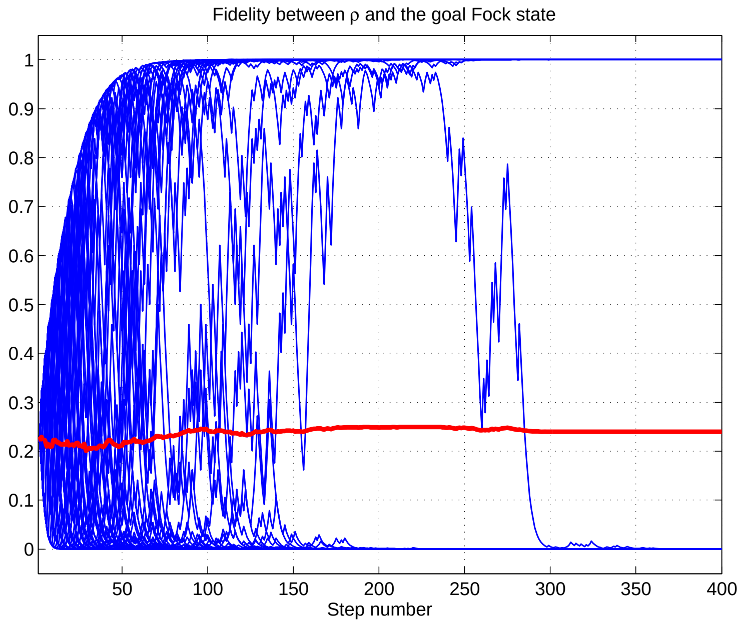

Figure 3.2: Fidelity with respect to the 3-photon state $\langle 3|\rho_k|3\rangle$ versus the number of passing atoms $k \in \{0, \ldots, 400\}$ for 100 realizations of the open-loop Markov process (3.6) (blue fine curves) starting from the same coherent state $\rho_0 = \mathbb{D}_{\sqrt{3}}(|0\rangle\langle 0|)$. The ensemble average over these realizations corresponds to the thick red curve.

3.2.2 Global convergence analysis

The following theorem underlies the observations made for simulations of Figure 3.2.

Theorem 3.1. *Consider the Markov process ρ_k obeying (3.6) with an initial condition $\rho_0 \in \mathcal{X}$ defined by (3.1). Then*

- *for any $n \in \{0, \ldots, n^{max}\}$, $\mathrm{Tr}\left(\rho_k |n\rangle\langle n|\right) = \langle n|\rho_k|n\rangle$ is a martingale,*
- *ρ_k converges with probability 1 to one of the $n^{max}+1$ Fock states $|n\rangle\langle n|$ with $n \in \{0, \ldots, n^{max}\}$,*
- *the probability to converge towards the Fock state $|n\rangle\langle n|$ is given by $\mathrm{Tr}\left(\rho_0 |n\rangle\langle n|\right) = \langle n|\rho_0|n\rangle$.*

Proof. Let us first prove that $\mathrm{Tr}\left(\rho_k |n\rangle\langle n|\right)$ is a martingale. Set $\xi = |n\rangle\langle n|$. We have

$$\begin{aligned}\mathbb{E}\left[\mathrm{Tr}\left(\xi\rho_{k+1}\right) \mid \rho_k\right] &= P_{g,k}\mathrm{Tr}\left(\xi\frac{M_g\rho_k M_g^\dagger}{P_{g,k}}\right) + P_{e,k}\mathrm{Tr}\left(\xi\frac{M_e\rho_k M_e^\dagger}{P_{e,k}}\right)\\ &= \mathrm{Tr}\left(\xi M_g\rho_k M_g^\dagger\right) + \mathrm{Tr}\left(\xi M_e\rho_k M_e^\dagger\right) = \mathrm{Tr}\left(\rho_k(M_g^\dagger\xi M_g + M_e^\dagger\xi M_e)\right).\end{aligned}$$

Since ξ commutes with M_g and M_e, and $M_g^\dagger M_g + M_e^\dagger M_e = \mathbb{1}$, we have $\mathbb{E}\left[\mathrm{Tr}\left(\xi\rho_{k+1}\right) \mid \rho_k\right] = \mathrm{Tr}\left(\xi\rho_k\right)$.

Consider now the following function:

$$\mathcal{V}_n(\rho) = f(\langle n|\rho|n\rangle),$$

where $f(x) = \frac{x+x^2}{2}$. Note that f is 1-convex, $f' \geq \frac{1}{2}$ on $[0,1]$, and it satisfies

$$\forall (x, y, \theta) \in [0,1], \quad \theta f(x) + (1-\theta) f(y) = \tfrac{\theta(1-\theta)}{2}(x-y)^2 + f(\theta x + (1-\theta) y). \tag{3.7}$$

The function f is increasing and convex and $\langle n|\rho_k|n\rangle$ is a martingale. Thus $\mathcal{V}_n(\rho_k)$ is a submartingale, $\mathbb{E}\left[\mathcal{V}_n(\rho_{k+1}) \mid \rho_k\right] \geq \mathcal{V}_n(\rho_k)$. Since

$$\langle n|\mathbb{M}_g(\rho)|n\rangle = \tfrac{\cos^2\varphi_n}{\mathrm{Tr}\left(M_g\rho M_g^\dagger\right)} \langle n|\rho|n\rangle, \quad \langle n|\mathbb{M}_e(\rho)|n\rangle = \tfrac{\sin^2\varphi_n}{\mathrm{Tr}\left(M_e\rho M_e^\dagger\right)} \langle n|\rho|n\rangle,$$

we have

$$\mathbb{E}\left[\mathcal{V}_n(\rho_{k+1}) \mid \rho_k\right] = \mathrm{Tr}\left(M_g\rho_k M_g^\dagger\right) f\left(\tfrac{\cos^2\varphi_n}{\mathrm{Tr}\left(M_g\rho_k M_g^\dagger\right)} \langle n|\rho_k|n\rangle\right) \\ + \mathrm{Tr}\left(M_e\rho_k M_e^\dagger\right) f\left(\tfrac{\sin^2\varphi_n}{\mathrm{Tr}\left(M_e\rho_k M_e^\dagger\right)} \langle n|\rho_k|n\rangle\right).$$

Then (3.7), together with

$$\theta = \mathrm{Tr}\left(M_g\rho_k M_g^\dagger\right), \ x = \tfrac{\cos^2\varphi_n}{\mathrm{Tr}\left(M_g\rho_k M_g^\dagger\right)} \langle n|\rho_k|n\rangle, \ y = \tfrac{\sin^2\varphi_n}{\mathrm{Tr}\left(M_e\rho_k M_e^\dagger\right)} \langle n|\rho_k|n\rangle,$$

yields to

$$\mathbb{E}\left[\mathcal{V}_n(\rho_{k+1}) \mid \rho_k\right] - \mathcal{V}_n(\rho_k) = \\ \frac{\mathrm{Tr}\left(M_g\rho_k M_g^\dagger\right)\mathrm{Tr}\left(M_e\rho_k M_e^\dagger\right)(\langle n|\rho_k|n\rangle)^2}{2}\left(\tfrac{\cos^2\varphi_n}{\mathrm{Tr}\left(M_g\rho_k M_g^\dagger\right)} - \tfrac{\sin^2\varphi_n}{\mathrm{Tr}\left(M_e\rho_k M_e^\dagger\right)}\right)^2.$$

We have also shown that $\mathbb{E}\left[\mathcal{V}_n(\rho_{k+1}) \mid \rho_k\right] = \mathcal{V}_n(\rho_k)$ implies that either $\langle n|\rho_k|n\rangle = 0$ or $\mathrm{Tr}\left(M_g\rho_k M_g^\dagger\right) = \cos^2\varphi_n$ (assumption M_g and M_e invertible is used here).

We apply now the invariance theorem established by Kushner [55] (recalled in Appendix A) for the Markov process ρ_k and the submartingale $\mathcal{V}_n(\rho_k)$. This theorem implies that the Markov process ρ_k converges in probability to the largest invariant subset of

$$\left\{\rho \in \mathcal{X} \mid \mathrm{Tr}\left(M_g\rho M_g^\dagger\right) = \cos^2\varphi_n \text{ or } \langle n|\rho|n\rangle = 0\right\}.$$

But the set $\{\rho \in \mathcal{X} \mid \langle n|\rho|n\rangle = 0\}$ is invariant. It thus remains to characterize the largest invariant subset denoted by $\mathcal{X}_n$ which is included in $\left\{\rho \in \mathcal{X} \mid \mathrm{Tr}\left(M_g\rho M_g^\dagger\right) = \cos^2\varphi_n\right\}$.

Take $\rho \in \mathcal{X}_n$. Invariance means that $\mathbb{M}_g(\rho)$ and $\mathbb{M}_e(\rho)$ belong to $\mathcal{X}_n$ (since M_g and M_e are invertible, the probabilities to detect $\mu = g$ or $\mu = e$ are strictly positive for any

$\rho \in \mathcal{X}$). Consequently, $\mathrm{Tr}\left(M_g \mathbb{M}_g(\rho) M_g^\dagger\right) = \mathrm{Tr}\left(M_g \rho M_g^\dagger\right) = \cos^2\varphi_n$. This means that $\mathrm{Tr}\left(M_g^4 \rho\right) = \mathrm{Tr}^2\left(M_g^2 \rho\right)$. By Cauchy-Schwartz inequality,

$$\mathrm{Tr}\left(M_g^4 \rho\right) = \mathrm{Tr}\left(M_g^4 \rho\right) \mathrm{Tr}\left(\rho\right) \geq \mathrm{Tr}^2\left(M_g^2 \rho\right)$$

with equality if and only if $M_g^4 \rho$ and ρ are co-linear. M_g^4 being non-degenerate, ρ is necessarily a projector over an eigenstate of M_g^4, i.e., $\rho = |m\rangle\langle m|$ for some $m \in \{0, \ldots, n^{\max}\}$. Since $\mathrm{Tr}\left(M_g \rho M_g^\dagger\right) = \cos^2\varphi_n > 0$, $m = n$ and thus $\mathcal{X}_n$ is reduced to $\{|n\rangle\langle n|\}$. Therefore the only possibilities for the ω-limit set are $\mathrm{Tr}\left(\rho\,|n\rangle\langle n|\right) = 0$ or 1 and

$$W_n(\rho_k) = \mathrm{Tr}\left(\rho_k\,|n\rangle\langle n|\right)\left(1 - \mathrm{Tr}\left(\rho_k\,|n\rangle\langle n|\right)\right) \overset{k\to\infty}{\longrightarrow} 0 \qquad \text{in probability.}$$

The convergence in probability together with the fact that $W_n(\rho_k)$ is a positive bounded ($W_n \in [0, 1]$) random process implies the convergence in expectation. Indeed

$$\begin{aligned}\limsup_{k\to\infty} \mathbb{E}\left[W_n(\rho_k)\right] &\leq \epsilon \limsup_{k\to\infty} \mathbb{P}\left(W_n(\rho_k) \leq \epsilon\right) + \limsup_{k\to\infty} \mathbb{P}\left(W_n(\rho_k) > \epsilon\right)\\ &\leq \epsilon + \limsup_{k\to\infty} \mathbb{P}\left(W_n(\rho_k) > \epsilon\right) \leq \epsilon,\end{aligned}$$

where for the last inequality, we have applied the convergence in probability of $W_n(\rho_k)$ towards 0. As the above inequality is valid for any $\epsilon > 0$, we have

$$\lim_{k\to\infty} \mathbb{E}\left[W_n(\rho_k)\right] = 0.$$

Furthermore, by the first part of Theorem 3.1, we know that $\mathrm{Tr}\left(\rho_k\,|n\rangle\langle n|\right)$ is a bounded martingale and therefore by Doob's first martingale convergence theorem (see Theorem A.1 in Appendix A), $\mathrm{Tr}\left(\rho_k\,|n\rangle\langle n|\right)$ converges almost surely towards a random variable $l_n^\infty \in [0, 1]$. This implies that $W_n(\rho_k)$ converges almost surely towards the random variable $l_n^\infty(1 - l_n^\infty) \in [0, 1]$. We apply now the dominated convergence theorem to get

$$\mathbb{E}\left[l_n^\infty(1 - l_n^\infty)\right] = \mathbb{E}\left[\lim_{k\to\infty} W_n(\rho_k)\right] = \lim_{k\to\infty} \mathbb{E}\left[W_n(\rho_k)\right] = 0.$$

This implies that $l_n^\infty(1 - l_n^\infty)$ vanishes almost surely and therefore

$$W_n(\rho_k) = \mathrm{Tr}\left(\rho_k\,|n\rangle\langle n|\right)\left(1 - \mathrm{Tr}\left(\rho_k\,|n\rangle\langle n|\right)\right) \overset{k\to\infty}{\longrightarrow} 0 \qquad \text{almost surely.}$$

Since we can repeat this same analysis for any choice of $n \in \{0, 1, \ldots, n^{\max}\}$, ρ_k converges almost surely to the set of Fock states

$$\{|n\rangle\langle n| \mid n = 0, 1, \ldots, n^{\max}\},$$

which ends the proof of the second part.

We have shown so far that the probability measure associated to the random variable ρ_k converges to the probability measure

$$\sum_{n=0}^{n^{\max}} p_n \delta(|n\rangle \langle n|),$$

where $\delta(|n\rangle \langle n|)$ denotes the Dirac distribution at $|n\rangle \langle n|$ and p_n is the probability of convergence towards $|n\rangle \langle n|$. In particular, we have

$$\mathbb{E}\left[\mathrm{Tr}\left(|n\rangle \langle n| \rho_k\right)\right] \overset{k\to\infty}{\longrightarrow} p_n.$$

But $\mathrm{Tr}\left(|n\rangle \langle n| \rho_k\right)$ is a martingale and $\mathbb{E}\left[\mathrm{Tr}\left(|n\rangle \langle n| \rho_k\right)\right] = \mathbb{E}\left[\mathrm{Tr}\left(|n\rangle \langle n| \rho_0\right)\right]$. Thus $p_n = \langle n| \rho_0 |n\rangle$, which ends the proof of the third and last part.

3.2.3 Local convergence rate

According to Theorem 3.1, the ω-limit set of the Markov process (3.6) is the discrete set of Fock states $\{|n\rangle \langle n|\}_{n\in\{0,\ldots,n^{\max}\}}$. We investigate here the local convergence rate around one of these Fock states denoted by $\bar{\rho} = |\bar{n}\rangle \langle \bar{n}|$ for some $\bar{n} \in \{0, \ldots, n^{\max}\}$.

Since $\mathbb{M}_g(\bar{\rho}) = \mathbb{M}_e(\bar{\rho}) = \bar{\rho}$, we can develop the dynamics (3.6) around the fixed point $\bar{\rho}$. We write $\rho = \bar{\rho} + \delta\rho$ with $\delta\rho$ small, Hermitian and with zero trace. Keeping only the first order terms in (3.6), we have

$$\delta\rho_{k+1} = \frac{M_{\mu_k}\delta\rho_k M_{\mu_k}^\dagger}{\mathrm{Tr}\left(M_{\mu_k}\bar{\rho}M_{\mu_k}^\dagger\right)} - \frac{\mathrm{Tr}\left(M_{\mu_k}\delta\rho_k M_{\mu_k}^\dagger\right)}{\mathrm{Tr}\left(M_{\mu_k}\bar{\rho}M_{\mu_k}^\dagger\right)}\bar{\rho}.$$

Thus, the linearized Markov process around the fixed point $\bar{\rho}$ reads

$$\delta\rho_{k+1} = A_{\mu_k}\delta\rho_k A_{\mu_k}^\dagger - \mathrm{Tr}\left(A_{\mu_k}\delta\rho_k A_{\mu_k}^\dagger\right)\bar{\rho}, \tag{3.8}$$

where the random matrices A_{μ_k} are given by :

- $A_g = \frac{M_g}{\cos\varphi_{\bar{n}}}$ with probability $P_g = \cos^2\varphi_{\bar{n}}$ and
- $A_e = \frac{M_e}{\sin\varphi_{\bar{n}}}$ with probability $P_e = \sin^2\varphi_{\bar{n}}$.

The following proposition shows that the convergence of the linearized dynamics is exponential (a crucial robustness indicator).

Proposition 3.1. *Consider the linear Markov chain (3.8) of state $\delta\rho$ belonging to the set of Hermitian matrices with zero trace. Then the largest Lyapunov exponent Λ is given by*

$$\Lambda = \max_{\substack{n \in \{0,\ldots,n^{max}\} \\ n \neq \bar{n}}} \left(\cos^2\varphi_{\bar{n}} \log\left(\frac{|\cos\varphi_n|}{|\cos\varphi_{\bar{n}}|}\right) + \sin^2\varphi_{\bar{n}} \log\left(\frac{|\sin\varphi_n|}{|\sin\varphi_{\bar{n}}|}\right)\right),$$

where $\varphi_n = \varphi_0 + n\vartheta$. In addition, Λ is strictly negative, $\Lambda < 0$.

Proof. Set $\delta\rho^{n_1,n_2} = \langle n_1|\delta\rho|n_2\rangle$ for any $n_1, n_2 \in \{0, \dots, n^{\max}\}$. Since $\mathrm{Tr}\,(\delta\rho_k) \equiv 0$, we exclude here the case $(n_1, n_2) = (\bar{n}, \bar{n})$ because $\delta\rho_k^{\bar{n},\bar{n}} = -\sum_{n\neq\bar{n}} \delta\rho_k^{n,n}$. Since A_e and A_g are diagonal matrices, we have

$$\delta\rho_{k+1}^{n_1,n_2} = a_{\mu_k}^{n_1,n_2}\delta\rho_k^{n_1,n_2}, \tag{3.9}$$

where $\mu_k = g$ (resp. $\mu_k = e$) with probability $\cos^2\varphi_{\bar{n}}$ (resp. $\sin^2\varphi_{\bar{n}}$) and where $a_g^{n_1,n_2} = \frac{\cos\varphi_{n_1}\cos\varphi_{n_2}}{\cos^2\varphi_{\bar{n}}}$ and $a_e^{n_1,n_2} = \frac{\sin\varphi_{n_1}\sin\varphi_{n_2}}{\sin^2\varphi_{\bar{n}}}$.

Denote by Λ^{n_1,n_2} the Lyapunov exponent of (3.9) for $(n_1, n_2) \neq (\bar{n}, \bar{n})$. By the law of large numbers, we know that $\frac{\log(|\prod_{l=0}^{l=k} a_{\mu_k}^{n_1,n_2}|)}{k+1}$ converges almost surely towards

$$\cos^2\varphi_{\bar{n}}\ \log(|a_g^{n_1,n_2}|) + \sin^2\varphi_{\bar{n}}\ \log(|a_e^{n_1,n_2}|).$$

Thus, we have

$$\Lambda^{n_1,n_2} = \cos^2\varphi_{\bar{n}}\left(\log\left(\tfrac{|\cos\varphi_{n_1}|}{|\cos\varphi_{\bar{n}}|}\right) + \log\left(\tfrac{|\cos\varphi_{n_2}|}{|\cos\varphi_{\bar{n}}|}\right)\right) + \sin^2\varphi_{\bar{n}}\left(\log\left(\tfrac{|\sin\varphi_{n_1}|}{|\sin\varphi_{\bar{n}}|}\right) + \log\left(\tfrac{|\sin\varphi_{n_2}|}{|\sin\varphi_{\bar{n}}|}\right)\right).$$

The function

$$\left[0, \frac{\pi}{2}\right] \ni \varphi \mapsto \left(\frac{\cos\varphi}{|\cos\varphi_{\bar{n}}|}\right)^{\cos^2\varphi_{\bar{n}}} \left(\frac{\sin\varphi}{|\sin\varphi_{\bar{n}}|}\right)^{\sin^2\varphi_{\bar{n}}}$$

increases strictly from 0 to 1 when φ goes from 0 to $\arcsin(|\sin\varphi_{\bar{n}}|)$, and decreases strictly from 1 to 0 when φ goes from $\arcsin(|\sin\varphi_{\bar{n}}|)$ to $\frac{\pi}{2}$. Since $(n_1, n_2) \neq (\bar{n}, \bar{n})$, we have $\Lambda^{n_1,n_2} < 0$. Denote $\Lambda^n = \Lambda^{n,\bar{n}}$ for $n \in \{0, \dots, n^{\max}\}$:

$$\Lambda^n = \cos^2\varphi_{\bar{n}}\log\left(\tfrac{|\cos\varphi_n|}{|\cos\varphi_{\bar{n}}|}\right) + \sin^2\varphi_{\bar{n}}\log\left(\tfrac{|\sin\varphi_n|}{|\sin\varphi_{\bar{n}}|}\right).$$

Since $(n_1, n_2) \neq (\bar{n}, \bar{n})$, we have $\Lambda^{n_1,n_2} \leq \max_{n\neq\bar{n}}\Lambda^n$ and $\Lambda = \max_{n\neq\bar{n}}\Lambda^n$ is strictly negative. □

3.3 Feedback stabilization with delays

Throughout this section we assume that we have access at each step k to the cavity state ρ_k. The goal is to design a causal feedback law that stabilizes globally the Markov chain (3.2) towards a goal Fock state $\bar{\rho} = |\bar{n}\rangle\langle\bar{n}|$ with $\bar{n}$ photon(s), $\bar{n} \in \{0, \dots, n^{\max}\}$. To be consistent with truncation to $n^{\max}$ photons, $\bar{n}$ has to be far from $n^{\max}$ (typically $\bar{n} = 3$ with $n^{\max} = 10$ in the simulations below).

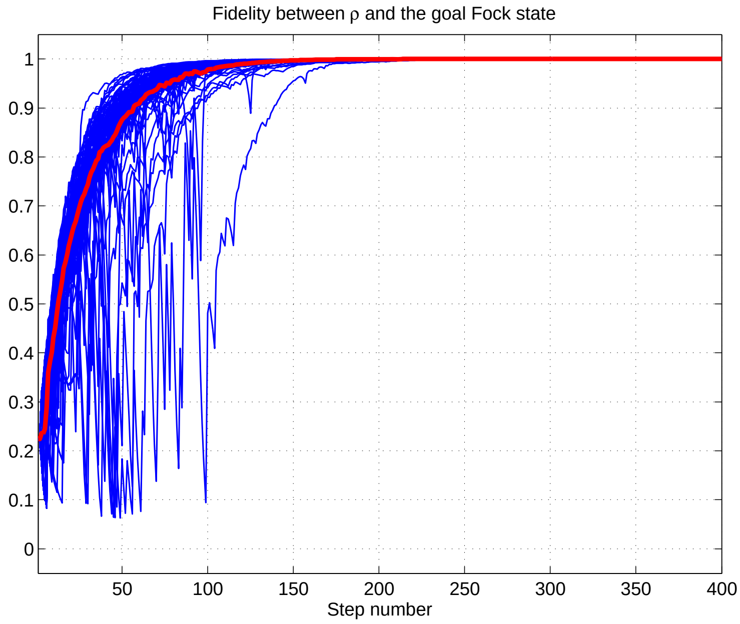

Figure 3.3: Fidelity with respect to the 3-photon state $\mathrm{Tr}\,(\rho_k\bar{\rho}) = \langle 3|\rho_k|3\rangle$ versus $k \in \{0,\ldots,400\}$ for 100 realizations of the closed-loop Markov process (3.2) with feedback (3.10) (blue fine curves) starting from the same state $\rho_0 = \mathbb{D}_{\sqrt{3}}(|0\rangle\langle 0|)$ (no delay, $\tau = 0$). The ensemble average over these realizations corresponds to the thick red curve.

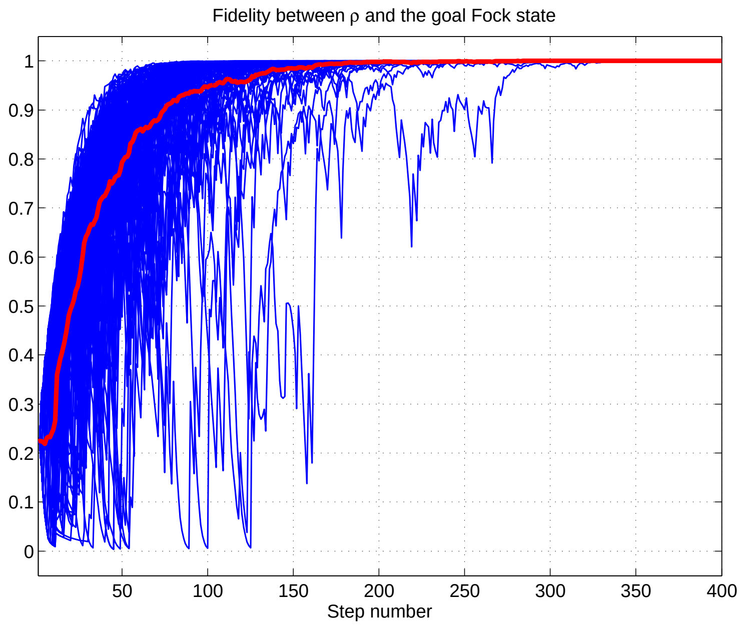

Figure 3.4: $\mathrm{Tr}\,(\rho_k\bar{\rho}) = \langle 3|\rho_k|3\rangle$ versus $k \in \{0,\ldots,400\}$ for 100 realizations of the closed-loop Markov process (3.12) with feedback (3.11) (blue fine curves) starting from the same state $\chi_0 = (\mathbb{D}_{\sqrt{3}}(|0\rangle\langle 0|), 0, \ldots, 0)$ and with 5-step delay ($\tau = 5$). The ensemble average over these realizations corresponds to the thick red curve.

3.3.1 Feedback scheme and closed-loop simulations

The feedback is based on the fact that, in open-loop when $\alpha_k \equiv 0$, $\mathrm{Tr}\left(\bar{\rho}\rho_k\right) = \langle \bar{n}|\rho_k|\bar{n}\rangle$ is a martingale. When $\tau = 0$, [68] proves global almost sure convergence of the following feedback law

$$\alpha_k = \begin{cases} \epsilon \mathrm{Tr}\left(\bar{\rho}\,[\rho_k, \mathbf{a}]\right) & \text{if } \mathrm{Tr}\left(\bar{\rho}\rho_k\right) \geq \varsigma, \quad \text{and} \\ \underset{|\alpha|\leq\bar{\alpha}}{\mathrm{argmax}}\left(\mathrm{Tr}\left(\bar{\rho}\,\mathbb{D}_\alpha(\rho_k)\right)\right) & \text{if } \mathrm{Tr}\left(\bar{\rho}\rho_k\right) < \varsigma, \end{cases} \tag{3.10}$$

for any $\bar{\alpha} > 0$ when $\epsilon, \varsigma > 0$ are small enough. This feedback law ensures that $\mathrm{Tr}\left(\bar{\rho}\rho_k\right)$ is a submartingale.

When $\tau > 0$, we cannot set $\alpha_{k-\tau} = \epsilon \mathrm{Tr}\left(\bar{\rho}\,[\rho_k, \mathbf{a}]\right)$ since α_k will depend on $\rho_{k+\tau}$ and the feedback law is not causal. In [36], this feedback law is made causal by replacing $\rho_{k+\tau}$ by its expectation value (average prediction) ρ_k^{pred} knowing ρ_k and the past controls $\alpha_{k-1}, \ldots,$ $\alpha_{k-\tau}$:

$$\rho_k^{\mathrm{pred}} = \mathbb{K}_{\alpha_{k-1}} \circ \ldots \circ \mathbb{K}_{\alpha_{k-\tau}}(\rho_k),$$

where the Kraus map $\mathbb{K}_\alpha$ is defined by (3.4).

We will thus consider here the following causal feedback based on an average compensation of the delay τ

$$\alpha_k = \begin{cases} \epsilon \mathrm{Tr}\left(\bar{\rho}\,[\rho_k^{\mathrm{pred}}, \mathbf{a}]\right) & \text{if } \mathrm{Tr}\left(\bar{\rho}\rho_k^{\mathrm{pred}}\right) \geq \varsigma, \quad \text{and} \\ \underset{|\alpha|\leq\bar{\alpha}}{\mathrm{argmax}}\left(\mathrm{Tr}\left(\bar{\rho}\,\mathbb{D}_\alpha(\rho_{g,k}^{\mathrm{pred}})\right)\mathrm{Tr}\left(\bar{\rho}\,\mathbb{D}_\alpha(\rho_{e,k}^{\mathrm{pred}})\right)\right) & \text{if } \mathrm{Tr}\left(\bar{\rho}\rho_k^{\mathrm{pred}}\right) < \varsigma, \end{cases} \tag{3.11}$$

with

$$\begin{cases} \rho_{g,k}^{\mathrm{pred}} = \mathbb{K}_{\alpha_{k-1}} \circ \ldots \circ \mathbb{K}_{\alpha_{k-\tau+1}}(M_g D_{\alpha_{k-\tau}} \rho_k D_{\alpha_{k-\tau}}^\dagger M_g^\dagger) \\ \rho_{e,k}^{\mathrm{pred}} = \mathbb{K}_{\alpha_{k-1}} \circ \ldots \circ \mathbb{K}_{\alpha_{k-\tau+1}}(M_e D_{\alpha_{k-\tau}} \rho_k D_{\alpha_{k-\tau}}^\dagger M_e^\dagger) \end{cases}$$

The closed-loop system, i.e., Markov chain (3.2), with the causal feedback (3.11), is still a Markov chain but with $(\rho_k, \alpha_{k-1}, \ldots, \alpha_{k-\tau})$ as state at step k. More precisely, denote by $\chi = (\rho, \beta_1, \ldots, \beta_\tau)$ this state where β_l stands for the control α delayed l steps. Then the state form of the closed-loop dynamics reads

$$\begin{cases} \rho_{k+1} &= \mathbb{M}_{\mu_k}(\mathbb{D}_{\beta_{\tau,k}}(\rho_k)) \\ \beta_{1,k+1} &= \alpha_k \\ \beta_{2,k+1} &= \beta_{1,k} \\ & \vdots \\ \beta_{\tau,k+1} &= \beta_{\tau-1,k}. \end{cases} \tag{3.12}$$

The control law defined by (3.11) corresponds to a static state feedback since

$$\begin{cases} \rho_k^{\mathrm{pred}} = \rho^{\mathrm{pred}}(\chi_k) = \mathbb{E}\left[\rho_{k+\tau} \mid \chi_k\right] = \mathbb{K}_{\beta_{1,k}} \circ \ldots \circ \mathbb{K}_{\beta_{\tau,k}}(\rho_k) \\ \rho_{g,k}^{\mathrm{pred}} = \rho_g^{\mathrm{pred}}(\chi_k) = \mathbb{K}_{\beta_{1,k}} \circ \ldots \circ \mathbb{K}_{\beta_{\tau-1,k}}(M_g D_{\beta_{\tau,k}} \rho_k D_{\beta_{\tau,k}}^\dagger M_g^\dagger) \\ \rho_{e,k}^{\mathrm{pred}} = \rho_e^{\mathrm{pred}}(\chi_k) = \mathbb{K}_{\beta_{1,k}} \circ \ldots \circ \mathbb{K}_{\beta_{\tau-1,k}}(M_e D_{\beta_{\tau,k}} \rho_k D_{\beta_{\tau,k}}^\dagger M_e^\dagger). \end{cases} \tag{3.13}$$

Note that $\rho_k^{\mathrm{pred}} = \rho_{g,k}^{\mathrm{pred}} + \rho_{e,k}^{\mathrm{pred}}$.

Simulations displayed on Figures 3.3 and 3.4 correspond to 100 realizations of the above closed-loop systems with $\tau = 0$ and $\tau = 5$. The goal state $\bar{\rho} = |3\rangle\langle 3|$ contains $\bar{n} = 3$ photons and $n^{\max}$, φ_0 and ϑ are those used for the open-loop simulations of Figure 3.2. Each realization starts with the same coherent state $\rho_0 = \mathbb{D}_{\sqrt{3}}(|0\rangle\langle 0|)$ and $\beta_{1,0} = \ldots = \beta_{\tau,0} = 0$. The feedback parameters appearing in (3.11) are as follows:

$$\epsilon = \tfrac{1}{2\bar{n}+1} = \tfrac{1}{7}, \quad \varsigma = \tfrac{1}{10}, \quad \bar{\alpha} = 1.$$

These simulations illustrate the influence of the delay τ on the average convergence speed: the longer the delay is the slower convergence speed becomes.

Remark 1. *The choice of the feedback law whenever* $\mathrm{Tr}\left(\bar{\rho}\rho_k^{pred}\right) < \varsigma$ *might seem complicated for real-time simulation issues. However, this choice is only technical. Actually, any non-zero constant feedback law seems to achieve the task here (see for instance the simulations of [36]). However, the convergence proof for such simplified control scheme is more complicated and is not considered in this chapter.*

3.3.2 Global convergence in closed-loop

The main result of this section is the following theorem.

Theorem 3.2. *Take the Markov chain (3.12) with the feedback (3.11) where* ρ_k^{pred}, $\rho_{g,k}^{pred}$ *and* $\rho_{e,k}^{pred}$ *are given by (3.13) with* $\bar{\alpha} > 0$. *Then, for small enough* $\epsilon > 0$ *and* $\varsigma > 0$, *the state* χ_k *converges almost surely towards* $\bar{\chi} = (\bar{\rho}, 0, \ldots, 0)$ *whatever the initial condition* $\chi_0 \in \mathcal{X} \times \mathbb{C}^\tau$ *is . Here the compact set* $\mathcal{X}$ *is defined as in (3.1).*

Proof. The proof is based on the Lyapunov-type function

$$\mathcal{V}(\chi) = f\left(\mathrm{Tr}\left(\bar{\rho}\rho^{\mathrm{pred}}\right)\right) \quad \text{with} \quad \rho^{\mathrm{pred}} = \mathbb{K}_{\beta_1} \circ \ldots \circ \mathbb{K}_{\beta_\tau}(\rho), \tag{3.14}$$

where $f(x) = \frac{x+x^2}{2}$ has already been used in the proof of Theorem 3.1. The proof relies on the following four lemmas:

- in Lemma 3.1, we prove an inequality showing that, for small enough ϵ, $\mathcal{V}(\chi)$ and $\mathrm{Tr}\left(\bar{\rho}\rho^{\mathrm{pred}}(\chi)\right)$ are submartingales within $\{\chi \mid \mathrm{Tr}\left(\bar{\rho}\rho^{\mathrm{pred}}\right) \geq \varsigma\}$;
- in Lemma 3.2, we show that for small enough ς, the trajectories starting within the set $\{\chi \mid \mathrm{Tr}\left(\bar{\rho}\rho^{\mathrm{pred}}\right) < \varsigma\}$ always reach in one step the set $\{\chi \mid \mathrm{Tr}\left(\bar{\rho}\rho^{\mathrm{pred}}\right) \geq 2\varsigma\}$;
- in Lemma 3.3, we show that the trajectories starting within the set $\{\chi \mid \mathrm{Tr}\left(\bar{\rho}\rho^{\mathrm{pred}}\right) \geq 2\varsigma\}$, will never hit the set $\{\chi \mid \mathrm{Tr}\left(\bar{\rho}\rho^{\mathrm{pred}}\right) < \varsigma\}$ with a uniformly non-zero probability $p > 0$;
- in Lemma 3.4, we combine the first step and the invariance principle due to Kushner, to prove that almost all trajectories remaining inside $\{\chi \mid \mathrm{Tr}\left(\bar{\rho}\rho^{\mathrm{pred}}\right) \geq \varsigma\}$ converge towards $\bar{\chi} = (\bar{\rho}, 0, \ldots, 0)$.

The combination of Lemmas 3.2, 3.3 and 3.4 shows then directly that χ_k converges almost surely towards $\bar{\chi}$. We detail now these four lemmas. □

Lemma 3.1. *For $\epsilon > 0$ small enough and for χ_k satisfying $\mathrm{Tr}\left(\bar{\rho}\rho^{pred}(\chi_k)\right) \geq \varsigma$,*

$$\mathbb{E}\left[\mathrm{Tr}\left(\bar{\rho}\rho^{pred}(\chi_{k+1})\right) \mid \chi_k\right] \geq \mathrm{Tr}\left(\bar{\rho}\rho^{pred}(\chi_k)\right) + \epsilon \left|\mathrm{Tr}\left(\bar{\rho}\ [\rho_k^{pred}, \mathbf{a}]\right)\right|^2,$$

and also

$$\begin{aligned}\mathbb{E}\left[\mathcal{V}(\chi_{k+1}) \mid \chi_k\right] \geq \mathcal{V}(\chi_k) + \tfrac{\epsilon}{2}\left|\mathrm{Tr}\left(\bar{\rho}\ [\rho_k^{pred}, \mathbf{a}]\right)\right|^2 \\ + \tfrac{P_{g,k}P_{e,k}}{2}\Big(\mathrm{Tr}\left(\bar{\rho}\ \mathbb{D}_{\alpha_k}\circ\mathbb{K}_{\beta_{1,k}}\circ\ldots\circ\mathbb{K}_{\beta_{\tau-1,k}}\circ\mathbb{M}_g\circ\mathbb{D}_{\beta_{\tau,k}}(\rho_k)\right) \\ - \mathrm{Tr}\left(\bar{\rho}\ \mathbb{D}_{\alpha_k}\circ\mathbb{K}_{\beta_{1,k}}\circ\ldots\circ\mathbb{K}_{\beta_{\tau-1,k}}\circ\mathbb{M}_e\circ\mathbb{D}_{\beta_{\tau,k}}(\rho_k)\right)\Big)^2. \quad (3.15)\end{aligned}$$

Proof. Since $M_g^\dagger M_g + M_e^\dagger M_e = \mathbb{1}$ and $[\bar{\rho}, M_g] = [\bar{\rho}, M_e] = 0$, we have

$$\begin{aligned}\mathrm{Tr}\left(\bar{\rho}\ \mathbb{K}_{\beta_{1,k+1}}\circ\mathbb{K}_{\beta_{2,k+1}}\circ\ldots\circ\mathbb{K}_{\beta_{\tau,k+1}}(\rho_{k+1})\right) = \\ \mathrm{Tr}\left(\bar{\rho}\ \mathbb{D}_{\beta_{1,k+1}}\circ\mathbb{K}_{\beta_{2,k+1}}\circ\ldots\circ\mathbb{K}_{\beta_{\tau,k+1}}(\rho_{k+1})\right).\end{aligned}$$

Also, we have

$$\begin{aligned}\mathbb{E}\left[f\left(\mathrm{Tr}\left(\bar{\rho}\ \mathbb{K}_{\beta_{1,k+1}}\circ\mathbb{K}_{\beta_{2,k+1}}\circ\ldots\circ\mathbb{K}_{\beta_{\tau,k+1}}(\rho_{k+1})\right)\right) \mid \chi_k\right] = \\ P_{g,k} f\left(\mathrm{Tr}\left(\bar{\rho}\ \mathbb{D}_{\alpha_k}\circ\mathbb{K}_{\beta_{1,k}}\circ\ldots\circ\mathbb{K}_{\beta_{\tau-1,k}}\circ\mathbb{M}_g\circ\mathbb{D}_{\beta_{\tau,k}}(\rho_k)\right)\right) + \\ P_{e,k} f\left(\mathrm{Tr}\left(\bar{\rho}\ \mathbb{D}_{\alpha_k}\circ\mathbb{K}_{\beta_{1,k}}\circ\ldots\circ\mathbb{K}_{\beta_{\tau-1,k}}\circ\mathbb{M}_e\circ\mathbb{D}_{\beta_{\tau,k}}(\rho_k)\right)\right).\end{aligned}$$

By (3.7) we find

$$\begin{aligned}\mathbb{E}\left[\mathcal{V}(\chi_{k+1}) \mid \chi_k\right] = f\left(\mathrm{Tr}\left(\bar{\rho}\ \mathbb{D}_{\alpha_k}\circ\mathbb{K}_{\beta_{1,k}}\circ\ldots\circ\mathbb{K}_{\beta_{\tau-1,k}}\circ\mathbb{K}_{\beta_{\tau,k}}(\rho_k)\right)\right) \\ + \tfrac{P_{g,k}P_{e,k}}{2}\left(\mathrm{Tr}\left(\bar{\rho}\ \mathbb{D}_{\alpha_k}\circ\mathbb{K}_{\beta_{1,k}}\circ\ldots\circ\mathbb{K}_{\beta_{\tau-1,k}}\left(\mathbb{M}_g\circ\mathbb{D}_{\beta_{\tau,k}}(\rho_k) - \mathbb{M}_e\circ\mathbb{D}_{\beta_{\tau,k}}(\rho_k)\right)\right)\right)^2\end{aligned}$$

Since $\rho^{\mathrm{pred}}(\chi_k) = \rho_k^{\mathrm{pred}} = \mathbb{K}_{\beta_{1,k}}\circ\ldots\circ\mathbb{K}_{\beta_{\tau-1,k}}\circ\mathbb{K}_{\beta_{\tau,k}}(\rho_k)$ we have

$$\mathrm{Tr}\left(\bar{\rho}\ \mathbb{D}_{\alpha_k}\circ\mathbb{K}_{\beta_{1,k}}\circ\ldots\circ\mathbb{K}_{\beta_{\tau-1,k}}\circ\mathbb{K}_{\beta_{\tau,k}}(\rho_k)\right) = \mathrm{Tr}\left(\bar{\rho}\ \mathbb{D}_{\alpha_k}(\rho_k^{\mathrm{pred}})\right).$$

For small α the Baker-Campbell-Hausdorff formula yields

$$\mathbb{D}_\alpha(\rho) = e^{\alpha\mathbf{a}^\dagger - \alpha^*\mathbf{a}}\ \rho\ e^{-(\alpha\mathbf{a}^\dagger - \alpha^*\mathbf{a})} = \rho + [\alpha\mathbf{a}^\dagger - \alpha^*\mathbf{a}, \rho] + O(|\alpha|^2).$$

Consequently

$$\mathrm{Tr}\left(\bar{\rho}\ \mathbb{D}_{\alpha_k}(\rho_k^{\mathrm{pred}})\right) = \mathrm{Tr}\left(\bar{\rho}\ \rho_k^{\mathrm{pred}}\right) + \mathrm{Tr}\left(\bar{\rho}\ [\alpha_k\mathbf{a}^\dagger - \alpha_k^*\mathbf{a}, \rho_k^{\mathrm{pred}}]\right) + O(|\alpha_k|^2).$$

Since $\alpha_k = \epsilon \mathrm{Tr}\left(\bar\rho[\rho_k^{\mathrm{pred}}, \mathbf{a}]\right)$, we get

$$\mathrm{Tr}\left(\bar\rho\ \mathbb{D}_{\alpha_k}(\rho_k^{\mathrm{pred}})\right) = \mathrm{Tr}\left(\bar\rho\ \rho_k^{\mathrm{pred}}\right) + 2\epsilon \left|\mathrm{Tr}\left(\bar\rho[\rho_k^{\mathrm{pred}}, \mathbf{a}]\right)\right|^2 + O(\epsilon^2).$$

We infer that for $\epsilon > 0$ small enough, uniformly in $\rho_k^{\mathrm{pred}} \in \mathcal{X}$,

$$\mathrm{Tr}\left(\bar\rho\ \mathbb{D}_{\alpha_k}(\rho_k^{\mathrm{pred}})\right) \geq \mathrm{Tr}\left(\bar\rho\ \rho_k^{\mathrm{pred}}\right) + \epsilon \left|\mathrm{Tr}\left(\bar\rho[\rho_k^{\mathrm{pred}}, \mathbf{a}]\right)\right|^2.$$

Using the fact that f is increasing and $f(x+y) \geq f(x) + y/2$ for any $x, y > 0$, we get

$$f\left(\mathrm{Tr}\left(\bar\rho\ \mathbb{D}_{\alpha_k}(\rho_k^{\mathrm{pred}})\right)\right) \geq f((\mathrm{Tr}\left(\bar\rho\ \rho_k^{\mathrm{pred}}\right)) + \tfrac{\epsilon}{2} \left|\mathrm{Tr}\left(\bar\rho[\rho_k^{\mathrm{pred}}, \mathbf{a}]\right)\right|^2,$$

which finishes the proof of the lemma. □

Lemma 3.2. *When $\varsigma > 0$ is small enough, any state χ_k satisfying the inequality $\mathrm{Tr}\left(\bar\rho\rho^{pred}(\chi_k)\right) < \varsigma$ yields a new state χ_{k+1} such that $\mathrm{Tr}\left(\bar\rho\rho^{pred}(\chi_{k+1})\right) \geq 2\varsigma$.*

Proof. Since M_g and M_e are invertible, there exists $\zeta \in]0,1[$ such that for any χ, $\mathrm{Tr}\left(\rho_g^{\mathrm{pred}}(\chi)\right) \geq \zeta$ and $\mathrm{Tr}\left(\rho_e^{\mathrm{pred}}(\chi)\right) \geq \zeta$ (ρ_g^{pred} and ρ_e^{pred} are defined in (3.13)). Denote by $\mathcal{X}_\zeta$ the compact set of Hermitian semi-definite positive matrices with trace in $[\zeta, 1]$: for any χ, $\rho_g^{\mathrm{pred}}(\chi)$ and $\rho_e^{\mathrm{pred}}(\chi)$ are in $\mathcal{X}_\zeta$. Let us first prove that for any $\rho_g, \rho_e \in \mathcal{X}_\zeta$,

$$\max_{|\alpha| \leq \bar\alpha} \left(\mathrm{Tr}\left(\bar\rho\ \mathbb{D}_\alpha(\rho_g)\right) \mathrm{Tr}\left(\bar\rho\ \mathbb{D}_\alpha(\rho_e)\right)\right) > 0. \tag{3.16}$$

If for some $\rho_g, \rho_e \in \mathcal{X}_\zeta$ the above maximum is zero, then for all $\alpha \in \mathbb{C}$ (analyticity of $\mathbb{D}_\alpha$ versus $\Re(\alpha)$ and $\Im(\alpha)$ [1]:

$$\mathrm{Tr}\left(\bar\rho\ \mathbb{D}_\alpha(\rho_g)\right) \mathrm{Tr}\left(\bar\rho\ \mathbb{D}_\alpha(\rho_e)\right) \equiv 0.$$

This implies that either $\mathrm{Tr}\left(\bar\rho\ \mathbb{D}_\alpha(\rho_g)\right) \equiv 0$ or $\mathrm{Tr}\left(\bar\rho\ \mathbb{D}_\alpha(\rho_e)\right) \equiv 0$ (if the product of two analytic functions is zero, one of them is zero). Take $\rho \in \mathcal{X}_\zeta$ such that $\mathrm{Tr}\left(\bar\rho\ \mathbb{D}_\alpha(\rho)\right) \equiv 0$. We can decompose ρ as a sum of projectors

$$\rho = \sum_{\nu=1}^{m} \lambda_\nu \left|\psi_\nu\right\rangle \left\langle\psi_\nu\right|,$$

where λ_ν are strictly positive eigenvalues, $\sum_\nu \lambda_\nu \in [\zeta, 1]$, and ψ_ν are the associated normalized eigenstates of ρ, $1 \leq m \leq n^{\max}$. Since $\mathrm{Tr}\left(\bar\rho\ \mathbb{D}_\alpha(\rho)\right) \equiv 0$ for all $\alpha \in \mathbb{C}$, we have for all ν, $\langle\psi_\nu|\ D_\alpha|\bar n\rangle = 0$. Fixing one $\nu \in \{1, \ldots, m\}$ and taking $\psi = \psi_\nu$ noting that $D_\alpha = \exp(\Re(\alpha)(\mathbf{a}^\dagger - \mathbf{a}) + \imath\Im(\alpha)(\mathbf{a}^\dagger + \mathbf{a})$ and deriving j times versus $\Re(\alpha)$ and $\Im(\alpha)$ around $\alpha = 0$, we get

$$\left\langle\psi \mid (\mathbf{a}^\dagger - \mathbf{a})^j |\bar n\right\rangle = \left\langle\psi \mid (\mathbf{a}^\dagger + \mathbf{a})^j |\bar n\right\rangle = 0 \qquad \forall j \geq 0.$$

With $j = 0$, we get, $\langle\psi\ |\bar n\rangle = 0$. With $j = 1$, since $\mathbf{a}^\dagger \left|\bar n\right\rangle = \sqrt{\bar n + 1} \left|\bar n + 1\right\rangle$ and $\mathbf{a} \left|\bar n\right\rangle = \sqrt{\bar n} \left|\bar n - 1\right\rangle$, we get $\langle\psi\ |\bar n - 1\rangle = \langle\psi\ |\bar n + 1\rangle = 0$. With $j = 2$, using the null Hermitian

[1]The notations $\Re(A)$ and $\Im(A)$ denote respectively the real and the imaginary part of A

products obtained for $j = 0$ and 1, we deduce that $\langle \psi \,|\bar{n}-2\rangle = \langle \psi \,|\bar{n}+2\rangle = 0$, since $\mathbf{a}\mathbf{a}^\dagger\,|\bar{n}\rangle$ and $\mathbf{a}^\dagger\mathbf{a}\,|\bar{n}\rangle$ are colinear to $|\bar{n}\rangle$. Similarly, for any j using the null Hermitian products obtained for $j' < j$, we deduce that $\langle \psi \,|\max(0, \bar{n}-j)\rangle = \langle \psi \,|\min(n^{\max}, \bar{n}+j)\rangle = 0$. Thus, for any n, $\langle \psi | n\rangle = 0$, $|\psi\rangle = 0$ and we get a contradiction. Thus (3.16) holds true for any $\rho_g, \rho_e \in \mathcal{X}_\zeta$.

The map F

$$F : (\rho_g, \rho_e) \mapsto F(\rho_g, \rho_e) = \max_{|\alpha|\leq\bar{\alpha}} \left(\mathrm{Tr}\left(\bar{\rho}\; \mathbb{D}_\alpha(\rho_g)\right) \mathrm{Tr}\left(\bar{\rho}\; \mathbb{D}_\alpha(\rho_e)\right)\right)$$

is continuous. We have proved that for all ρ_g, ρ_e in the compact set $\mathcal{X}_\zeta$, $F(\rho_g, \rho_e) > 0$. Therefore, there exists $\delta > 0$ such that $F(\rho_g, \rho_e) \geq \delta$ for any $\rho_g, \rho_e \in \mathcal{X}_\zeta$. Take $\tilde{\alpha}$ an argument of the maximum,

$$\mathrm{Tr}\left(\bar{\rho}\; \mathbb{D}_{\tilde{\alpha}}(\rho_g)\right) \mathrm{Tr}\left(\bar{\rho}\; \mathbb{D}_{\tilde{\alpha}}(\rho_e)\right) = \max_{|\alpha|\leq\bar{\alpha}} \left(\mathrm{Tr}\left(\bar{\rho}\; \mathbb{D}_\alpha(\rho_g)\right) \mathrm{Tr}\left(\bar{\rho}\; \mathbb{D}_\alpha(\rho_e)\right)\right) \geq \delta.$$

Since $\mathrm{Tr}\left(\bar{\rho}\; \mathbb{D}_{\tilde{\alpha}}(\rho_g)\right) \leq 1$ (Cauchy-Schwartz inequality for the Frobenius product) and $\mathrm{Tr}\left(\bar{\rho}\; \mathbb{D}_{\tilde{\alpha}}(\rho_e)\right) \leq 1$, we have $\mathrm{Tr}\left(\bar{\rho}\; \mathbb{D}_{\tilde{\alpha}}(\rho_g)\right) \geq \delta$ and $\mathrm{Tr}\left(\bar{\rho}\; \mathbb{D}_{\tilde{\alpha}}(\rho_e)\right) \geq \delta$.

Take now $\varsigma < \frac{\delta}{2}$ and χ_k such that $\mathrm{Tr}\left(\bar{\rho}\rho^{\mathrm{pred}}(\chi_k)\right) \leq \varsigma$. According to (3.11), α_k is chosen as an argument of

$$\max_{|\alpha|\leq\bar{\alpha}} \left(\mathrm{Tr}\left(\bar{\rho}\; \mathbb{D}_\alpha(\rho^{\mathrm{pred}}_{g,k})\right) \mathrm{Tr}\left(\bar{\rho}\; \mathbb{D}_\alpha(\rho^{\mathrm{pred}}_{e,k})\right)\right),$$

where $\rho^{\mathrm{pred}}_{g,k}, \rho^{\mathrm{pred}}_{e,k} \in \mathcal{X}_\zeta$. Thus, $\mathrm{Tr}\left(\bar{\rho}\; \mathbb{D}_{\alpha_k}(\rho^{\mathrm{pred}}_{g,k})\right) \geq \delta$ and $\mathrm{Tr}\left(\bar{\rho}\; \mathbb{D}_{\alpha_k}(\rho^{\mathrm{pred}}_{e,k})\right) \geq \delta$. But either $\rho^{\mathrm{pred}}_{k+1} = \frac{1}{P_{g,k}}\mathbb{K}_{\alpha_k}(\rho^{\mathrm{pred}}_{g,k})$ or $\rho^{\mathrm{pred}}_{k+1} = \frac{1}{P_{e,k}}\mathbb{K}_{\alpha_k}(\rho^{\mathrm{pred}}_{e,k})$ where $0 < P_{g,k}, P_{e,k} < 1$. Since we have the identity $\mathrm{Tr}\left(\bar{\rho}\mathbb{K}_\alpha(\rho)\right) = \mathrm{Tr}\left(\bar{\rho}\mathbb{D}_\alpha(\rho)\right)$, because $\bar{\rho}$ commutes with M_g and M_e, we conclude that $\mathrm{Tr}\left(\bar{\rho}\; \rho^{\mathrm{pred}}(\chi_{k+1})\right) \geq \delta \geq 2\varsigma$. □

Lemma 3.3. *Initializing the Markov process χ_k within the set $\{\chi \mid \mathrm{Tr}\left(\bar{\rho}\rho^{pred}(\chi)\right) \geq 2\varsigma\}$, with some probability*

$$p > \frac{\varsigma}{1-\varsigma} > 0,$$

χ_k will never hit the set $\{\chi \mid \mathrm{Tr}\left(\bar{\rho}\rho^{pred}(\chi)\right) < \varsigma\}$

Proof. We know from Lemma 3.1 that the process $1 - \mathrm{Tr}\left(\bar{\rho}\rho^{\mathrm{pred}}(\chi)\right)$ is a supermartingale in the set $\{\chi \mid \mathrm{Tr}\left(\bar{\rho}\rho^{\mathrm{pred}}(\chi)\right) \geq \varsigma\}$. Therefore, one only needs to use the Doob's inequality recalled in Appendix A:

$$\mathbb{P}\left(\sup_{0\leq k<\infty}\left(1 - \mathrm{Tr}\left(\bar{\rho}\rho^{\mathrm{pred}}(\chi_k)\right)\right) > 1-\varsigma\right) > \frac{1 - \mathrm{Tr}\left(\bar{\rho}\rho^{\mathrm{pred}}(\chi_0)\right)}{1-\varsigma} \geq \frac{1-2\varsigma}{1-\varsigma},$$

which shows the lemma for $p > 1 - \frac{1-2\varsigma}{1-\varsigma} = \frac{\varsigma}{1-\varsigma}$. □

Lemma 3.4. *Sample paths χ_k remaining in the set $\{\mathrm{Tr}\left(\bar{\rho}\rho^{pred}(\chi)\right) \geq \varsigma\}$ converges almost surely to $\bar{\chi}$ as $k \to \infty$.*

Proof. We apply first the Kushner's invariance theorem to the Markov process χ_k with the submartingale function $\mathcal{V}(\chi_k)$. It ensures convergence in probability towards $\mathcal{I}$ the largest invariant set attached to this submartingale (see Appendix A). Let us prove that $\mathcal{I}$ is reduced to $\{\bar{\chi}\}$.

By inequality (3.15), if $(\rho, \beta_1, \ldots, \beta_\tau) = \chi$ belongs to $\mathcal{I}$, then $\mathrm{Tr}\left(\bar{\rho}\left[\rho^{\mathrm{pred}}(\chi), \mathbf{a}\right]\right) = 0$, i.e., $\alpha \equiv 0$ and also

$$\mathrm{Tr}\left(\bar{\rho}\, \mathbb{D}_\alpha \circ \mathbb{K}_{\beta_1} \circ \ldots \circ \mathbb{K}_{\beta_{\tau-1}} \circ \mathbb{M}_g \circ \mathbb{D}_{\beta_\tau}(\rho)\right) = \mathrm{Tr}\left(\bar{\rho}\, \mathbb{D}_\alpha \circ \mathbb{K}_{\beta_1} \circ \ldots \circ \mathbb{K}_{\beta_{\tau-1}} \circ \mathbb{M}_e \circ \mathbb{D}_{\beta_\tau}(\rho)\right).$$

Invariance associated $\alpha \equiv 0$ implies that $\beta_1 = \ldots = \beta_\tau = 0$. Hence the above equality reads

$$\mathrm{Tr}\left(\bar{\rho}\, \mathbb{M}_g(\rho)\right) = \mathrm{Tr}\left(\bar{\rho}\, \mathbb{M}_e(\rho)\right),$$

where we have used the fact that, for any $\varrho \in \mathcal{X}$, $\mathrm{Tr}\left(\bar{\rho}\, \mathbb{K}_0(\varrho)\right) = \mathrm{Tr}\left(\bar{\rho}\, \mathbb{D}_0(\varrho)\right) = \mathrm{Tr}\left(\bar{\rho}\varrho\right)$. Then ρ satisfies the equation

$$\mathrm{Tr}\left(\bar{\rho} M_g \rho M_g^\dagger\right) \mathrm{Tr}\left(M_e \rho M_e^\dagger\right) = \mathrm{Tr}\left(\bar{\rho} M_e \rho M_e^\dagger\right) \mathrm{Tr}\left(M_g \rho M_g^\dagger\right)$$

that reads,

$$\cos^2\varphi_{\bar{n}} \mathrm{Tr}\left(M_e \rho M_e^\dagger\right) = \sin^2\varphi_{\bar{n}} \mathrm{Tr}\left(M_g \rho M_g^\dagger\right),$$

since $M_g^\dagger \bar{\rho} M_g = \cos^2\varphi_{\bar{n}}\ \bar{\rho}$, $M_e^\dagger \bar{\rho} M_e = \sin^2\varphi_{\bar{n}}\ \bar{\rho}$ and $\mathrm{Tr}\left(\bar{\rho}\rho\right) > 0$.

Since $\mathrm{Tr}\left(M_e \rho M_e^\dagger\right) + \mathrm{Tr}\left(M_g \rho M_g^\dagger\right) = 1$, we recover $\mathrm{Tr}\left(M_g \rho M_g^\dagger\right) = \cos^2\varphi_{\bar{n}}$ the same condition as the one appearing at the end of the proof of Theorem 3.1. Similar invariance arguments combined with $\mathrm{Tr}\left(\bar{\rho}\rho\right) > 0$ imply $\rho = \bar{\rho}$. We conclude that $\mathcal{I}$ is reduced to $\{\bar{\chi}\}$.

Consider now the event $P_{\geq\varsigma} = \{\forall k \geq 0,\ \mathrm{Tr}\left(\bar{\rho}\rho^{\mathrm{pred}}(\chi_k)\right) \geq \varsigma\}$. Convergence of χ_k in probability towards $\bar{\chi}$ means that

$$\forall \delta > 0, \quad \lim_{k\to\infty} \mathbb{P}\left(\|\chi_k - \bar{\chi}\| > \delta \mid P_{\geq\varsigma}\right) = 0,$$

where $\|\cdot\|$ is any norm on the χ-space. The continuity of $\chi \mapsto \mathrm{Tr}\left(\bar{\rho}\rho^{\mathrm{pred}}(\chi)\right)$ implies that, $\forall \delta > 0$,

$$\lim_{k\to\infty} \mathbb{P}\left(\mathrm{Tr}\left(\bar{\rho}\rho^{\mathrm{pred}}(\chi_k)\right) < 1 - \delta \mid P_{\geq\varsigma}\right) = 0.$$

Since $0 \leq \mathrm{Tr}\left(\bar{\rho}\rho^{\mathrm{pred}}(\chi)\right) \leq 1$, we have

$$1 \geq \mathbb{E}\left[\mathrm{Tr}\left(\bar{\rho}\rho^{\mathrm{pred}}(\chi_k)\right) \mid P_{\geq\varsigma}\right] \geq (1-\delta)\mathbb{P}\left(1 - \delta \leq \mathrm{Tr}\left(\bar{\rho}\rho^{\mathrm{pred}}(\chi_k)\right) \mid P_{\geq\varsigma}\right).$$

Thus

$$1 \geq \mathbb{E}\left[\mathrm{Tr}\left(\bar{\rho}\rho^{\mathrm{pred}}(\chi_k)\right) \mid P_{\geq\varsigma}\right] \geq 1 - \delta - \mathbb{P}\left(\mathrm{Tr}\left(\bar{\rho}\rho^{\mathrm{pred}}(\chi_k)\right) < 1 - \delta \mid P_{\geq\varsigma}\right).$$

Consequently, $\forall \delta > 0$, $\limsup_{k\to\infty} \mathbb{E}\left[\mathrm{Tr}\left(\bar{\rho}\rho^{\mathrm{pred}}(\chi_k)\right) \mid P_{\geq\varsigma}\right] \geq 1 - \delta$, i.e.,

$$\lim_{k\to\infty} \mathbb{E}\left[\mathrm{Tr}\left(\bar{\rho}\rho^{\mathrm{pred}}(\chi_k)\right) \mid P_{\geq\varsigma}\right] = 1.$$

The process $\mathrm{Tr}\left(\bar{\rho}\rho^{\mathrm{pred}}(\chi_k)\right)$ is a bounded submartingale and so, by Theorem A.1 of Appendix A, we know that it converges for almost all trajectories remaining in the set $\{\mathrm{Tr}\left(\bar{\rho}\rho^{\mathrm{pred}}(\chi)\right) \geq \varsigma\}$. Denoting the limit random variable by fid_∞, we have by the dominated convergence theorem

$$\mathbb{E}\left[\mathrm{fid}_\infty\right] = \mathbb{E}\left[\lim_{k\to\infty} \mathrm{Tr}\left(\bar{\rho}\rho^{\mathrm{pred}}(\chi_k)\right) \mid P_{\geq\varsigma}\right] = \lim_{k\to\infty} \mathbb{E}\left[\mathrm{Tr}\left(\bar{\rho}\rho^{\mathrm{pred}}(\chi_k)\right) \mid P_{\geq\varsigma}\right] = 1.$$

This trivially proves that $\mathrm{fid}_\infty \equiv 1$ almost surely and finishes the proof of the lemma. $\square$

$\square$

3.3.3 Convergence rate around the target state

Around the target state $\bar{\chi} = (\bar{\rho}, 0, \ldots, 0)$, the closed-loop dynamics reads

$$\begin{aligned}
\rho_{k+1} &= \mathbb{M}_{\mu_k}(\mathbb{D}_{\beta_{\tau,k}}(\rho_k)) \\
\beta_{1,k+1} &= \epsilon \mathrm{Tr}\left([\mathbf{a}, \bar{\rho}]\ \mathbb{K}_{\beta_{1,k}} \circ \ldots \circ \mathbb{K}_{\beta_{\tau,k}}(\rho_k)\right) \\
\beta_{2,k+1} &= \beta_{1,k} \\
&\vdots \\
\beta_{\tau,k+1} &= \beta_{\tau-1,k}.
\end{aligned}$$

Set $\chi = \bar{\chi} + \delta\chi$ with $\delta\chi = (\delta\rho, \delta\beta_1, \ldots, \delta\beta_\tau)$ small. Computations based on

$$\begin{aligned}
&\mathbb{D}_{\delta\beta}(\bar{\rho}) = \bar{\rho} + \left(\delta\beta[\mathbf{a}^\dagger, \bar{\rho}] - \delta\beta^*[\mathbf{a}, \bar{\rho}]\right) + O(|\delta\beta|^2), \\
&\mathbb{K}_{\delta\beta}(\bar{\rho}) = \mathbb{K}_0(\bar{\rho}) + \cos\vartheta\left(\delta\beta[\mathbf{a}^\dagger, \bar{\rho}] - \delta\beta^*[\mathbf{a}, \bar{\rho}]\right) + O(|\delta\beta|^2), \\
&\mathbb{K}_0(\bar{\rho}) = \bar{\rho}, \quad \mathbb{K}_0([\mathbf{a}^\dagger, \bar{\rho}]) = \cos\vartheta\ [\mathbf{a}^\dagger, \bar{\rho}], \quad \mathbb{K}_0([\mathbf{a}, \bar{\rho}]) = \cos\vartheta\ [\mathbf{a}, \bar{\rho}], \\
&\mathrm{Tr}\left([\mathbf{a}, \bar{\rho}][\mathbf{a}^\dagger, \bar{\rho}]\right) = -(2\bar{n}+1) \quad \text{and} \quad \mathrm{Tr}\left([\mathbf{a}, \bar{\rho}]^2\right) = 0,
\end{aligned}$$

yield the following linearized closed-loop system

$$\begin{aligned}
\delta\rho_{k+1} &= A_{\mu_k}\left(\delta\rho_k + \delta\beta_{\tau,k}[\mathbf{a}^\dagger, \bar{\rho}] - \delta\beta^*_{\tau,k}[\mathbf{a}, \bar{\rho}]\right) A^\dagger_{\mu_k} - \mathrm{Tr}\left(A_{\mu_k}\delta\rho_k A^\dagger_{\mu_k}\right)\bar{\rho} \\
\delta\beta_{1,k+1} &= -\epsilon(2\bar{n}+1)\left(\textstyle\sum_{j=1}^{\tau} \cos^j\vartheta\ \delta\beta_{j,k}\right) + \epsilon\cos^\tau\vartheta\ \mathrm{Tr}\left(\delta\rho_k[\mathbf{a}, \bar{\rho}]\right) \\
\delta\beta_{2,k+1} &= \delta\beta_{1,k} \\
&\vdots \\
\delta\beta_{\tau,k+1} &= \delta\beta_{\tau-1,k}
\end{aligned} \tag{3.17}$$

where $\mu_k \in \{g, e\}$, and the random matrices A_{μ_k} are given by $A_g = \frac{M_g}{\cos\varphi_{\bar{n}}}$ with probability $P_g = \cos^2\varphi_{\bar{n}}$ and $A_e = \frac{M_e}{\sin\varphi_{\bar{n}}}$ with probability $P_e = \sin^2\varphi_{\bar{n}}$.

Set $\delta\rho_k^{n_1,n_2} = \langle n_1|\delta\rho_k|n_2\rangle$ for any $n_1, n_2 \in \{0, \ldots, n^{\max}\}$. Since $\mathrm{Tr}\left(\delta\rho_k\right) \equiv 0$, we exclude here the case $(n_1, n_2) = (\bar{n}, \bar{n})$ because $\delta\rho_k^{\bar{n},\bar{n}} = -\sum_{n\neq\bar{n}} \delta\rho_k^{n,n}$. When (n_1, n_2) does not belong to $\{(\bar n-1,\bar n),(\bar n+1,\bar n),(\bar n,\bar n-1),(\bar n,\bar n+1)\}$, we recover the open-loop linearized dynamics (3.9):

$$\delta\rho_{k+1}^{n_1,n_2}=a_{\mu_k}^{n_1,n_2}\delta\rho_k^{n_1,n_2},$$

where $\mu_k=g$ (resp. $\mu_k=e$) with probability $\cos^2\varphi_{\bar n}$ (resp. $\sin^2\varphi_{\bar n}$), and where $a_g^{n_1,n_2}=\frac{\cos\varphi_{n_1}\cos\varphi_{n_2}}{\cos^2\varphi_{\bar n}}$ and $a_e^{n_1,n_2}=\frac{\sin\varphi_{n_1}\sin\varphi_{n_2}}{\sin^2\varphi_{\bar n}}$. A direct adaptation of the proof of Proposition 3.1 shows that the largest Lyapounov exponent Λ_0 of this dynamics is strictly negative and is given by

$$\Lambda_0=\max_{\substack{n\in\{0,\dots,n^{\max}\}\\ n\neq\bar n-1,\bar n,\bar n+1}}\left(\cos^2\varphi_{\bar n}\log\left(\tfrac{|\cos\varphi_n|}{|\cos\varphi_{\bar n}|}\right)+\sin^2\varphi_{\bar n}\log\left(\tfrac{|\sin\varphi_n|}{|\sin\varphi_{\bar n}|}\right)\right).$$

For $(n_1,n_2)\in\{(\bar n-1,\bar n),(\bar n+1,\bar n),(\bar n,\bar n-1),(\bar n,\bar n+1)\}$, we just have to consider $x=\delta\rho^{\bar n,\bar n-1}$ and $y=\delta\rho^{\bar n+1,\bar n}$, since $\delta\rho$ is Hermitian. Set $z_{j,k}=\delta\beta_{j,k}$. We deduce from (3.17) that the process $X_k=(x_k,y_k,z_{1,k},\dots,z_{\tau,k})$ is governed by

$$\begin{aligned}
x_{k+1}&=a_{\mu_k}(x_k-\sqrt{\bar n}z_{\tau,k})\\
y_{k+1}&=b_{\mu_k}(y_k+\sqrt{\bar n+1}z_{\tau,k})\\
z_{1,k+1}&=-\epsilon(2\bar n+1)\left(\textstyle\sum_{j=1}^{\tau}\cos^j\vartheta z_{j,k}\right)+\epsilon\cos^\tau\vartheta\left(\sqrt{\bar n}x_k-\sqrt{\bar n+1}y_k\right)\\
z_{2,k+1}&=z_{1,k}\\
&\vdots\\
z_{\tau,k+1}&=z_{\tau-1,k},
\end{aligned}\tag{3.18}$$

where $\mu_k=g$ (resp. $\mu_k=e$) with probability $\cos^2\varphi_{\bar n}$ (resp. $\sin^2\varphi_{\bar n}$), and

$$a_g=\tfrac{\cos\varphi_{\bar n-1}}{\cos\varphi_{\bar n}},\quad a_e=\tfrac{\sin\varphi_{\bar n-1}}{\sin\varphi_{\bar n}},\quad b_g=\tfrac{\cos\varphi_{\bar n+1}}{\cos\varphi_{\bar n}},\quad b_e=\tfrac{\sin\varphi_{\bar n+1}}{\sin\varphi_{\bar n}}.$$

Take $s>0$ to be defined later, set $\Sigma=|\cos\vartheta|\in]0,1[$ and consider

$$\mathcal{P}(X)=|x|+|y|+s\left(|z_1|+\Sigma|z_2|+\ldots+\Sigma^{\tau-1}|z_\tau|\right).$$

A direct computation exploiting (3.18) yields

$$\begin{aligned}
\mathbb{E}\left[\mathcal{P}(X_{k+1})\mid X_k\right]=&\,\Sigma|x_k-\sqrt{\bar n}z_{\tau,k}|+\Sigma|y_k+\sqrt{\bar n+1}z_{\tau,k}|\\
&+\Sigma s\left(|z_{1,k}|+\Sigma|z_{2,k}|+\ldots+\Sigma^{\tau-2}|z_{\tau-1,k}|\right)\\
&+\epsilon s\left|-(2\bar n+1)\left(\sum_{j=1}^{\tau}\cos^j\vartheta z_{j,k}\right)+\cos^\tau\vartheta\left(\sqrt{\bar n}x_k-\sqrt{\bar n+1}y_k\right)\right|.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\mathbb{E}\left[\mathcal{P}(X_{k+1})\mid X_k\right]\le(\Sigma+\epsilon s\Sigma^\tau\sqrt{\bar n+1})(|x_k|+|y_k|)\\
&\quad+\Sigma(1+\epsilon(2\bar n+1))s\left(|z_{1,k}|+\ldots+\Sigma^{\tau-2}|z_{\tau-1,k}|+\tfrac{\Sigma^{1-\tau}(\sqrt{\bar n}+\sqrt{\bar n+1})+\epsilon s(2\bar n+1)}{(1+\epsilon(2\bar n+1))s}\,\Sigma^{\tau-1}|z_{\tau,k}|\right).
\end{aligned}$$

For the value $s = \frac{\sqrt{\bar{n}}+\sqrt{\bar{n}+1}}{\Sigma^{\tau-1}}$, we get

$$\mathbb{E}\left[\mathcal{P}(X_{k+1}) \mid X_k\right] \leq \Sigma(1 + 2\epsilon(\bar{n}+1))\ \mathcal{P}(X_k).$$

Because $\Sigma < 1$, for $\epsilon > 0$ small enough ($\epsilon < \frac{1-\Sigma}{2(\bar{n}+1)}$), the norm $\mathcal{P}(X_k)$ is a supermartingale converging exponentially almost surely towards zero. Therefore the largest Lyapunov exponent of the linear Markov chain (3.18) is strictly negative. To conclude, we have proved the following proposition:

Proposition 3.2. *Consider the linear Markov chain (3.17). For small enough $\epsilon > 0$, its largest Lyapunov exponent is strictly negative.*

Remark 2. *We remark that the delay τ can be taken arbitrary large. This can be observed through the proof of Theorem 3.2 that our stability result does not depend on the number of delays. However, it is not necessarily interesting for the experimentalists to take a large delay which decreases the convergence rate.*

3.4 Quantum filter and separation principle

The feedback law (3.11) requires the knowledge of $(\rho_k, \beta_{1,k}, \ldots, \beta_{\tau,k})$. When the measurement process is fully efficient and the jump model (3.2) admits no error, the Markov system (3.12) represents a natural choice for the quantum filter to estimate the value of ρ. Indeed, we define the estimator $\chi_k^e = (\rho_k^e, \beta_{1,k}, \ldots, \beta_{\tau,k})$ satisfying the dynamics

$$\begin{cases} \rho_{k+1}^e &= \mathbb{M}_{\mu_k}(\mathbb{D}_{\beta_{\tau,k}}(\rho_k^e)) \\ \beta_{1,k+1} &= \alpha_k \\ \beta_{2,k+1} &= \beta_{1,k} \\ & \vdots \\ \beta_{\tau,k+1} &= \beta_{\tau-1,k}. \end{cases} \tag{3.19}$$

Note that, similarly to any observer-controller structure, the jump result, $\mu_k = g$ or e, is the output of the physical system (3.2) but the feedback control α_k is a function of the estimator ρ^e. Indeed, α_k is defined as in (3.11):

$$\alpha_k = \begin{cases} \epsilon \mathrm{Tr}\left(\bar{\rho}\left[\rho_k^{\text{pred},e}, \mathbf{a}\right]\right) & \text{if } \mathrm{Tr}\left(\bar{\rho}\rho_k^{\text{pred},e}\right) \geq \varsigma \\ \underset{|\alpha|\leq\bar{\alpha}}{\operatorname{argmax}} \left(\mathrm{Tr}\left(\bar{\rho}\ \mathbb{D}_\alpha(\rho_{g,k}^{\text{pred},e})\right) \mathrm{Tr}\left(\bar{\rho}\ \mathbb{D}_\alpha(\rho_{e,k}^{\text{pred},e})\right)\right) & \text{if } \mathrm{Tr}\left(\bar{\rho}\rho_k^{\text{pred},e}\right) < \varsigma \end{cases} \tag{3.20}$$

where the predictor's state $\rho_k^{\text{pred},e}$ is defined as follows:

$$\begin{cases} \rho_k^{\text{pred},e} = \mathbb{K}_{\alpha_{k-1}} \circ \ldots \circ \mathbb{K}_{\alpha_{k-\tau}}(\rho_k^e) \\ \rho_{g,k}^{\text{pred},e} = \mathbb{K}_{\alpha_{k-1}} \circ \ldots \circ \mathbb{K}_{\alpha_{k-\tau+1}}(M_g D_{\alpha_{k-\tau}} \rho_k^e D_{\alpha_{k-\tau}}^\dagger M_g^\dagger) \\ \rho_{e,k}^{\text{pred},e} = \mathbb{K}_{\alpha_{k-1}} \circ \ldots \circ \mathbb{K}_{\alpha_{k-\tau+1}}(M_e D_{\alpha_{k-\tau}} \rho_k^e D_{\alpha_{k-\tau}}^\dagger M_e^\dagger). \end{cases}$$

We will see through this section that, even if do not have any a priori knowledge of the initial state of the physical system, the choice of the feedback law through the above quantum filter can ensure the convergence of the system towards the desired Fock state. Indeed, we prove a semi-global robustness of the feedback scheme with respect to the choice of the initial state of the quantum filter.

Before going through the details of this robustness analysis, let us illustrate it through some numerical simulations in the next subsection.

3.4.1 Quantum filter and closed-loop simulations

In the simulations of Figure 3.5, we assume no a priori knowledge on the initial state of the system. Therefore, we initialize the filter equation at the maximally mixed state $\rho^e_0 = \frac{1}{n^{\max}+1}\mathbb{1}_{(n^{\max}+1)\times(n^{\max}+1)}$. Computing the feedback control through the above quantum filter and injecting it to the physical system modeled by (3.2), the fidelity (with respect to the target Fock state) of the closed-loop trajectories of the physical system are illustrated in the first plot of Figure 3.5. The second plot of this figure illustrate the Frobenius distance between the estimator ρ^e and the physical state ρ. As one can easily see, one still has the convergence of the quantum filter and the physical system to the desired Fock state (here $|3\rangle\langle 3|$).

Through these simulations, we have considered the same measurement and control parameters as those of Section 3.3. The system is initialized at the coherent state $\rho_0 = \mathbb{D}_{\sqrt{3}}(|0\rangle\langle 0|)$ while the quantum filter is initialized at $\rho^e_0 = \frac{1}{n^{\max}+1}\mathbb{1}_{(n^{\max}+1)\times(n^{\max}+1)}$.

Through the next subsection, we establish a sort of separation principle implying this semi-global robustness of the closed-loop system with respect to the initial state of the filter equation. Also through the short Subsection 3.4.3 we provide a heuristic analysis of the local convergence rate of the filter equation around the target Fock state.

3.4.2 A quantum separation principle

We consider the joint system-observer dynamics defined for the state $\Xi_k = (\rho_k, \rho^e_k, \beta_{1,k}, \ldots, \beta_{\tau,k})$:

$$\begin{cases} \rho_{k+1} &= \mathbb{M}_{\mu_k}(\mathbb{D}_{\beta_{\tau,k}}(\rho_k)) \\ \rho^e_{k+1} &= \mathbb{M}_{\mu_k}(\mathbb{D}_{\beta_{\tau,k}}(\rho^e_k)) \\ \beta_{1,k+1} &= \alpha_k \\ \beta_{2,k+1} &= \beta_{1,k} \\ & \vdots \\ \beta_{\tau,k+1} &= \beta_{\tau-1,k}. \end{cases} \tag{3.21}$$

We have the following result, a quantum version of the separation principle, ensuring the asymptotic stability of observer/controller from the stability of the observer and of the controller separatly.

Theorem 3.3. *Consider any closed-loop system of the form (3.21), where the feedback law α_k is a function of the quantum filter: $\alpha_k = g(\rho^e_k, \beta_{1,k}, \ldots, \beta_{\tau,k})$. Assume moreover that, whenever $\rho^e_0 = \rho_0$ (so that the quantum filter coincides with the closed-loop*

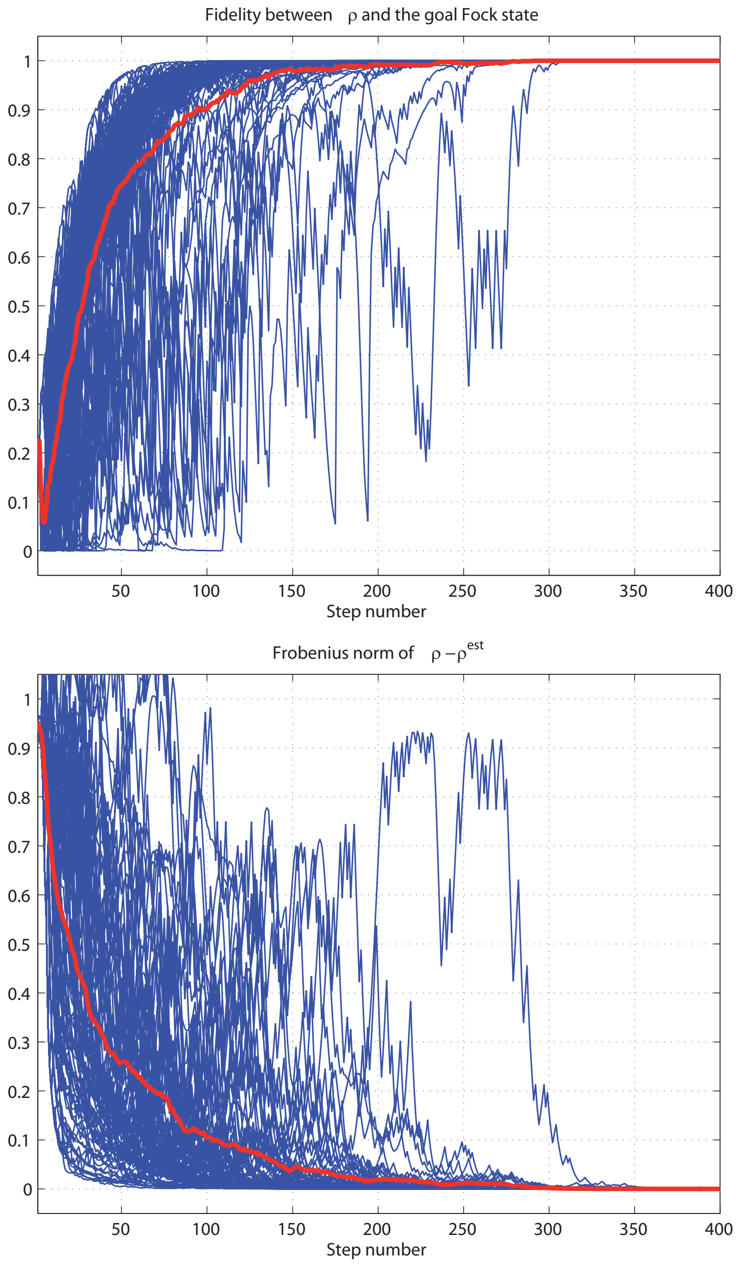

Figure 3.5: (First plot) $\mathrm{Tr}\left(\rho_k \bar{\rho}\right) = \langle 3|\rho_k|3\rangle$ versus $k \in \{0, \ldots, 400\}$ for 100 realizations of the closed-loop Markov process (3.2) with feedback (3.20) based on the quantum filter (3.19) starting from the same state $\chi_0^e = (\frac{1}{n^{\max}+1}\mathbb{1}_{(n^{\max}+1)\times(n^{\max}+1)}, 0, \ldots, 0)$ with 5-step delay ($\tau = 5$). The initial state of the physical system ρ_0 is given by $\mathbb{D}_{\sqrt{3}}(|0\rangle\langle 0|)$. The ensemble average over these realizations corresponds to the thick red curve; (Second plot) The Frobenius distance between the estimator ρ^e and ρ ($\sqrt{\mathrm{Tr}\left((\rho - \rho^e)^2\right)}$) for 100 realizations. The ensemble average over these realizations corresponds to the thick red curve.

dynamics ([3.12](#))*), the closed-loop system ρ_k and the estimate state ρ_k^e converge almost surely towards a fixed pure state $\bar{\rho}$. Then, for any choice of the initial state ρ_0^e, such that $\ker\rho_0^e \subset \ker\rho_0$, the trajectories of the system ρ_k and also the estimate state ρ_k^e converge almost surely towards the same pure state: $\rho_k \to \bar{\rho}$ and $\rho_k^e \to \bar{\rho}$.*

Remark 3. *One only needs to choose $\rho_0^e = \frac{1}{n^{max}+1}\mathbb{1}_{(n^{max}+1)\times(n^{max}+1)}$, so that the assumption $\ker\rho_0^e \subset \ker\rho_0$ is satisfied for any ρ_0.*

Proof. The basic idea is based on the fact that $\mathbb{E}\left[\mathrm{Tr}\left(\rho_k\bar{\rho}\right) \mid \rho_0, \rho_0^e\right]$ (where we take the expectation over all jump realizations) depends linearly on ρ_0 even though we are applying a feedback control. Indeed, the feedback law α_k depends only on the historic of the quantum jumps as well as on the initialization of the quantum filter ρ_0^e. Therefore, we can write

$$\beta_{k,\tau} = \alpha_{k-\tau} = \alpha(\rho_0^e, \mu_0, \ldots, \mu_{k-\tau-1}),$$

where $\{\mu_j\}_{j=0}^{k-1}$ denotes the sequence of k first jumps. Finally, through simple computations, we have

$$\mathbb{E}\left[\mathrm{Tr}\left(\rho_k\bar{\rho}\right) \mid \rho_0, \rho_0^e\right] = \sum_{\mu_0,\ldots,\mu_{k-1}} \mathrm{Tr}\left(\widetilde{\mathbb{M}}_{\mu_{k-1}} \circ \mathbb{D}_{\beta_{k,\tau}} \circ \ldots \circ \widetilde{\mathbb{M}}_{\mu_0} \circ \mathbb{D}_{\beta_{0,\tau}}\rho_0\right),$$

where

$$\widetilde{\mathbb{M}}_\mu\rho = M_\mu\rho M_\mu^\dagger.$$

So, we easily have the linearity of $\mathbb{E}\left[\mathrm{Tr}\left(\rho_k\bar{\rho}\right) \mid \rho_0, \rho_0^e\right]$ with respect to ρ_0.

At this point, we apply the assumption $\ker\rho_0^e \subset \ker\rho_0$ and therefore, we can find a constant $\gamma > 0$ and a well-defined density matrix ρ_0^c in $\mathcal{X}$ such that

$$\rho_0^e = \gamma\rho_0 + (1-\gamma)\rho_0^c.$$

Now, considering the system ([3.21](#)) initialized at the state $(\rho_0^e, \rho_0^e, 0, \ldots, 0)$, we have by the assumptions of the theorem and by applying the dominated convergence theorem:

$$\lim_{k\to\infty} \mathbb{E}\left[\mathrm{Tr}\left(\rho_k\bar{\rho}\right) \mid \rho_0^e, \rho_0^e\right] = 1.$$

By the linearity of $\mathbb{E}\left[\mathrm{Tr}\left(\rho_k\bar{\rho}\right) \mid \rho_0, \rho_0^e\right]$ with respect to ρ_0, we get

$$\mathbb{E}\left[\mathrm{Tr}\left(\rho_k\bar{\rho}\right) \mid \rho_0^e, \rho_0^e\right] = \gamma\mathbb{E}\left[\mathrm{Tr}\left(\rho_k\bar{\rho}\right) \mid \rho_0, \rho_0^e\right] + (1-\gamma)\mathbb{E}\left[\mathrm{Tr}\left(\rho_k\bar{\rho}\right) \mid \rho_0^c, \rho_0^e\right],$$

and since both $\mathbb{E}\left[\mathrm{Tr}\left(\rho_k\bar{\rho}\right) \mid \rho_0, \rho_0^e\right]$ and $\mathbb{E}\left[\mathrm{Tr}\left(\rho_k\bar{\rho}\right) \mid \rho_0^c, \rho_0^e\right]$ are less than or equal to one, we necessarily obtain that both of them converge to 1:

$$\lim_{k\to\infty} \mathbb{E}\left[\mathrm{Tr}\left(\rho_k\bar{\rho}\right) \mid \rho_0, \rho_0^e\right] = 1.$$

This implies the almost sure convergence of the physical system towards the pure state $\bar{\rho}$. We infer that $\mathbb{E}\left[\mathrm{Tr}\left(\rho_k^e\bar{\rho}|\rho_0, \rho_0^e\right)\right]$ depends linearly on ρ_0. Thus,

$$\mathbb{E}\left[\mathrm{Tr}\left(\rho_k^e\bar{\rho}\right)|\rho_0^e, \rho_0^e\right] = \gamma\mathbb{E}\left[\mathrm{Tr}\left(\rho_k^e\bar{\rho}\right)|\rho_0, \rho_0^e\right] + (1-\gamma)\mathbb{E}\left[\mathrm{Tr}\left(\rho_k^e\bar{\rho}\right)|\rho_0^c, \rho_0^e\right].$$

Using the fact that when $\rho_0 = \rho_0^e$, we have $\rho_k = \rho_k^e$ for all k, we conclude similarly that ρ_k^e converges almost surely towards $\bar{\rho}$ even if ρ_0 and ρ_0^e do not coincide. $\square$

3.4.3 Local convergence rate for the quantum filter

Let us linearize the system-observer dynamics (3.21) around the equilibrium state $\bar{\Xi} = (\bar{\rho}, \bar{\rho}, 0, \dots, 0)$. Set $\Xi = \bar{\Xi} + \delta\Xi$ with $\delta\Xi = (\delta\rho, \delta\rho^e, \delta\beta_1, \dots, \delta\beta_\tau)$ small, $\delta\rho$ and $\delta\rho^e$ Hermitian and of trace 0. We have the following dynamics for the linearized system (adaptation of (3.17)):

$$\begin{aligned}
\delta\rho_{k+1} &= A_{\mu_k}\left(\delta\rho_k + \delta\beta_{\tau,k}[\mathbf{a}^\dagger, \bar{\rho}] - \delta\beta^*_{\tau,k}[\mathbf{a}, \bar{\rho}]\right) A^\dagger_{\mu_k} - \mathrm{Tr}\left(A_{\mu_k}\delta\rho_k A^\dagger_{\mu_k}\right)\bar{\rho} \\
\delta\rho^e_{k+1} &= A_{\mu_k}\left(\delta\rho^e_k + \delta\beta_{\tau,k}[\mathbf{a}^\dagger, \bar{\rho}] - \delta\beta^*_{\tau,k}[\mathbf{a}, \bar{\rho}]\right) A^\dagger_{\mu_k} - \mathrm{Tr}\left(A_{\mu_k}\delta\rho^e_k A^\dagger_{\mu_k}\right)\bar{\rho} \\
\delta\beta_{1,k+1} &= -\epsilon(2\bar{n}+1)\left(\textstyle\sum_{j=1}^{\tau}\cos^j\vartheta\ \delta\beta_{j,k}\right) + \epsilon\cos^\tau\vartheta\ \mathrm{Tr}\left(\delta\rho^e_k[\mathbf{a}, \bar{\rho}]\right) \\
\delta\beta_{2,k+1} &= \delta\beta_{1,k} \\
&\ \ \vdots \\
\delta\beta_{\tau,k+1} &= \delta\beta_{\tau-1,k},
\end{aligned} \tag{3.22}$$

where $\mu_k \in \{g, e\}$, and the random matrices A_{μ_k} are given by $A_g = \frac{M_g}{\cos\varphi_{\bar{n}}}$ with probability $P_g = \cos^2\varphi_{\bar{n}}$ and $A_e = \frac{M_e}{\sin\varphi_{\bar{n}}}$ with probability $P_e = \sin^2\varphi_{\bar{n}}$.

At this point, we note that by considering $\widetilde{\delta\rho}_k = \delta\rho^e_k - \delta\rho_k$, we have the following simple dynamics:

$$\widetilde{\delta\rho}_{k+1} = A_{\mu_k}\widetilde{\delta\rho}_k A^\dagger_{\mu_k} - \mathrm{Tr}\left(A_{\mu_k}\widetilde{\delta\rho}_k A^\dagger_{\mu_k}\right)\bar{\rho}.$$

Indeed, as the same control laws are applied to the quantum filter and the physical system, the difference between $\delta\rho^e_k$ and $\delta\rho_k$ follows the same dynamics as the linearized open-loop system (3.8). But, we know by Proposition 3.1 that this linear system admits strictly negative Lyapunov exponents. This triangular structure, together with the convergence rate analysis of the closed-loop system in Proposition 3.2, yields the following proposition whose detailed proof is left to the reader:

Proposition 3.3. *Consider the linear Markov chain (3.22). For small enough $\epsilon > 0$, its largest Lyapunov exponent is strictly negative.*

3.5 Conclusion

We have analyzed a measurement-based feedback control allowing to stabilize globally and deterministically a desired Fock state. In this feedback design, we have taken into account the important delay between the measurement process and the feedback injection. This delay has been compensated by a stochastic version of a Kalman-type predictor in the quantum filtering equation.

In fact, the measurement process of the experimental setup [32] admits some other imperfections. These imperfections can, essentially, be resumed to the following ones:

1. The atom-detector is not fully efficient and it can miss some of the atoms, we denote the detector's efficiency by $\eta_d \in (0, 1]$ (η_d is about 0.8 for the LKB experimental setup);

2. The atom-detector is not fault-free and the result of the measurement (atom in the state g or e) can be inter-changed, we denote the fault rate by $\eta_f \in [0, \frac{1}{2}]$ (a fault rate of about 10%);

3. The atom preparation process is itself a stochastic process following a Poisson law, and therefore the measurement pulses can be empty of atom, we note the occupancy rate by $\eta_a \in (0, 1]$ (a pulse occupation rate of about 40%).

The knowledge of all these rates can help us to adapt the quantum filter by taking into account these imperfections. This has been done in [36], by considering the Bayesian law and by providing numerical evidence of the efficiency of such feedback algorithms assuming all these imperfections.

Indeed, whatever the initial state is, we can achieve three results: the atom is in $|g\rangle$, the atom is in $|e\rangle$, there are no atom detected. In each case, with the notations of the probabilities given in above, we can express the conditional evolution of the density operator.

- **Atom is detected in $|\mathbf{g}\rangle$.** Either the atom is in the state $|e\rangle$ and the detector has made a mistake by detecting it in the state $|g\rangle$ which arrives with the following probability

$$P_g^f = \frac{\eta_f P_e}{\eta_f P_e + (1-\eta_f)P_g}\ ^{2},$$

  where $P_g = \mathrm{Tr}\left(M_g \rho M_g^\dagger\right)$ and $P_e = \mathrm{Tr}\left(M_e \rho M_e^\dagger\right)$. Or the atom is really in the state $|g\rangle$, this happens with probability $1-P_g^f$. So , the conditional evolution of the density matrix (as our knowledge on the cavity state conditioned on the measurement result) at step k is given as follows:

$$\rho_{k+1} = P_g^f \mathbb{M}_e(\rho_k) + (1-P_g^f)\mathbb{M}_g(\rho_k) = \frac{\eta_f M_e \rho_k M_e^\dagger + (1-\eta_f) M_g \rho_k M_g^\dagger}{\eta_f P_e + (1-\eta_f)P_g},$$

  which is also in accordance with the recursive formula (2.7).

- **Atom is detected in $|\mathbf{e}\rangle$.** In the same way, the conditional evolution of the density matrix in this situation is given by the following

$$\rho_{k+1} = \frac{\eta_f M_g \rho_k M_g^\dagger + (1-\eta_f) M_e \rho_k M_e^\dagger}{\eta_f P_g + (1-\eta_f)P_e},$$

  which can also be obtained by the formula (2.7).

[2] where we have used the Bayes' rule: $P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A)+P(B|A^c)P(A^c)}$, with A^c the complementary of A.

- **No atom is detected.** Either the pulse has been empty, which arrives with probability P_{na} given below, or there has been an atom which has not been detected by the detector, this arrives with probability $1 - P_{na}$. The probability P_{na} can be computed through the Bayes' rule:

$$P_{na} = \frac{1 - \eta_a}{\eta_a(1 - \eta_d) + (1 - \eta_a)} = \frac{1 - \eta_a}{1 - \eta_a \eta_d}.$$

Then with probability P_{na}, the density matrix remains unchanged. The complementary situation corresponds to an undetected atom. In this case, the evolution of the density matrix is described by the Kraus operator $\mathbb{K}(\rho_k) := M_g \rho_k M_g^\dagger + M_e \rho_k M_e^\dagger$. We can now describe the conditional evolution as follows

$$\rho_{k+1} = P_{na}\rho_k + (1 - P_{na})(M_g \rho_k M_g^\dagger + M_e \rho_k M_e^\dagger) = \frac{(1 - \eta_a)\rho_k + \eta_a(1 - \eta_d)(M_g \rho_k M_g^\dagger + M_e \rho_k M_e^\dagger)}{1 - \eta_a \eta_d}.$$

This formula can be also derived from the recursive equation (2.7).

In the next chapter, we will consider the measurement imperfections and their impacts on the control of discrete-time systems subject to QND measurements.

In [36], the stabilizing feedback of this chapter which takes into account all the imperfections of the experimental setup has been tested in simulations. It has also been tested experimentally but the obtained fidelity was slightly less than the one given for the feedback law developed in the next chapter.

Chapter 4

Feedback stabilization under discrete-time QND measurements

Contents

Dans le chapitre précédent, nous avons étudié une rétroaction basée sur la mesure (qui prend en compte les retards) pour stabiliser un état arbitraire de nombre de photons dans la boite de photons. La méthode est basée sur l'utilisation de la fonction de Lyapunov naturelle $1 - \langle \bar{n}| \rho |\bar{n}\rangle$. Mais dans la mise en œvre expérimentale [88, 87], la rétroaction d'état a été améliorée par un meilleur choix sur la fonction de Lyapunov. Le but de ce chapitre est de présenter les méthodes mathématiques sous-jacentes à de telles constructions de Lyapunov améliorées qui simplifient l'analyse de la convergence, car l'application des principes d'invariance de Kushner n'est pas nécessaire.

Dans ce chapitre, une rétroaction d'état est appliquée aux systèmes génériques en temps discret et de dimensions finies dans le but de préparer et stabiliser le système sur un état cible prédéfini. Nous considérons deux types de dynamiques. La première décrit l'évolution des systèmes quantiques soumis à des mesures QND et contrôlés par une évolution unitaire réglable entre deux mesures successives QND. Les dynamiques de ces systèmes sont décrites par des modèles de Markov séparable (i.e., par l'application de deux opérateurs successifs: d'abord l'opérateur de mesure puis l'opérateur de contrôle unitaire). En boucle ouverte, ces mesures QND fournissent un outil de préparation non-déterministe exploitant l'action de retour de la mesure sur l'état quantique. Nous proposons ici une méthode systématique basée sur la théorie des graphes élémentaires et inversion des matrices de Laplace pour construire des fonctions de Lyapunov stricte. Cela donne une loi de rétroaction appropriée qui stabilise globalement le système vers un état cible choisi parmi ceux qui sont stables en boucle ouverte; ceci rend cette préparation déterministe en boucle fermée. La deuxième dynamique décrit l'évolution des systèmes quantiques contrôlés soumis à des POVM qui dépendent de la commande u. En boucle ouverte, les mesures quantiques sont supposée être non-destructives (QND). Ce processus de Markov admet donc un ensemble d'états purs associés à une base orthonormée de l'espace de Hilbert sous-jacent. Ces états stationnaires fournissent des martingales qui sont essentielles pour la caractérisation de la stabilité en boucle ouverte : sous des hypothèses simples et suggestives, presque toutes les trajectoires convergent vers un de ces états stationnaires ; la probabilité de converger vers un état stationnaire est donnée par sa distance avec l'état initial. À partir de ces martingales en boucle ouverte, nous construisons une sur-martingale dont les paramètres sont donnés par l'inversion d'une matrice Metzler caractérisant l'impact de la commande en sur les opérateurs de Kraus définissant le processus de Markov.

Dans la Section 4.1, nous considérons les systèmes dont les évolutions sont décrites par des modèles de Markov séparables. La Section 4.2 est consacrée aux systèmes dont les dynamiques sont données par des modèles de Markov non-séparables. Dans la Section 4.3, nous reconsidérons des chaînes de Markov non-séparable contrôlées, mais en présence de certains retards dans le processus de mesure. Dans cette section, nous démontrons, en utilisant un filtre de prédiction quantique, que le schéma proposé dans la section précédante fonctionne même en présence de retards. Dans la Section 4.4, nous supposons qu'il existe des retards dans le processus de mesure et que les mesures sont imparfaites. Notre résultat principal est donné dans le Théorème 4.10 où la convergence en boucle fermée est prouvée en présence de retards et d'imperfection de mesure. La Section 4.5 est consacrée à la mise en œvre expérimentale qui a été réalisée au Laboratoire Kastler Brossel (LKB) à l'École Normale Supérieure (ENS) de Paris. Les simulations en boucle fermée et les données expérimentales complémentaires à celles rapportées dans [87, 88] sont présentées. Les résultats de la Section 4.1 sont principalement basés sur [5]. Les Sections 4.3, 4.4 et 4.5 sont directement dérivées de notre travail [6].

In the previous chapter, we have studied a measurement-based feedback (which takes into account delays) to stabilize an arbitrary photon-number state in the photon box. The method was based on using the natural Lyapunov function $1 - \langle \bar{n} | \rho | \bar{n} \rangle$. But in the experimental implementation [88, 87], the state feedback has been improved by a better choice of Lyapunov function. The goal of this chapter is to present the mathematical methods underlying such improved Lyapunov designs which simplifies the convergence analysis, since the application of Kushner's invariance principles [55] is not necessary.

In this chapter, a state feedback scheme is applied to generic discrete-time finite-dimensional quantum systems, in the aim of preparing and stabilizing the system at some pre-specified goal state. We consider two types of dynamics. The first one describes the evolution of quantum systems subject to QND measurements and controlled by an adjustable unitary evolution between two successive QND measures. The dynamics of such systems are described by separable Markov models (i.e., by applying two successive operators: first the measurement operator and then the unitary control operator). In open-loop, such QND measurements provide a non-deterministic preparation tool exploiting the back-action of the measurement on the quantum state. We propose here a systematic method based on elementary graph theory and inversion of Laplacian matrices to construct strict control-Lyapunov functions. This yields an appropriate feedback law that stabilizes globally the system towards a chosen target state among the open-loop stable ones, and that makes in closed-loop this preparation deterministic. The second dynamics describes the evolution of controlled quantum systems subject to POVMs that depend on the control u. In open-loop, the measurements are assumed to be quantum non-demolition (QND). This Markov process admits thus a set of stationary pure states associated to an orthonormal basis of the underlying Hilbert space. These stationary states provide martingales that are crucial to characterizing the open-loop stability: under simple and suggestive assumptions, almost all trajectories converge to one of these stationary states; the probability to converge to a stationary state is given by its overlap with the initial quantum state. From these open-loop martingales, we construct a supermartingale whose parameters are given by inverting a Metzler matrix characterizing the impact of the control input on the Kraus operators defining the Markov process.

In Section 4.1, we consider the systems whose evolutions are described by separable Markov models. Section 4.2 is devoted to the systems whose dynamics are given by non-separable Markov models. In Section 4.3, we reconsider the non-separable controlled Markov chain but in presence of some delays in the measurement process. In this section, we demonstrate, using a predictive quantum filter, that the scheme proposed in the previous section works even in presence of delays. In Section 4.4, we suppose that there exist some delays in the measurement process and also that the measurements are imperfect. Our main result is given in Theorem 4.10 where closed-loop convergence is proved in presence of delays and measurement imperfections. Section 4.5 is devoted to the experimental implementation that has been carried on in Laboratoire Kastler Brossel (LKB) at Ecole Normale Supérieure (ENS) de Paris. Closed-loop simulations and experimental data complementary to those reported in [87, 88] are presented. The results of Section 4.1 are mainly based on [5]. Sections 4.3, 4.4 and 4.5 are directly derived from our work [6].

4.1 Strict control-Lyapunov functions for separable Markov models

4.1.1 The separable Markov model

We consider a finite-dimensional quantum system (so the underlying Hilbert space $\mathcal{H} = \mathbb{C}^d$ being of dimension $d > 0$) measured through a generalized measurement procedure placed discretely in time. Between two measurements, the system undergoes a unitary evolution depending on a scalar control input $u \in \mathbb{R}$. The dynamics of such discrete-time quantum systems is described by a non-linear controlled Markov chain whose structure is derived from quantum physics. We just sketch here this structure with a mathematical viewpoint, and note that a tutorial physical exposure can be found in [44].

The system state is described by the density operator ρ belonging to $\mathcal{D}$ the space of positive, Hermitian matrices of trace one:

$$\mathcal{D} := \{\rho \in \mathbb{C}^{d\times d} \mid \quad \rho = \rho^\dagger, \quad \mathrm{Tr}\,(\rho) = 1, \quad \rho \geq 0\}.$$

The generalized measurement procedure admits $m > 0$ different discrete values $\mu \in \{1, \ldots, m\}$: to each measurement outcome μ is attached a Kraus operator described by a matrix $M_\mu \in \mathbb{C}^{d\times d}$. The Kraus operators $(M_\mu)_{\mu\in\{1,\ldots,m\}}$ satisfy the constraint $\sum_{\mu=1}^{m} M_\mu^\dagger M_\mu = \mathbb{1}$, where $\mathbb{1}$ is the identity matrix. In general the M_μ are not necessarily Hermitian. The controlled evolution between two measures U_u is defined by the unitary operator $\exp(-iuH) = U_u$, where H is a Hermitian operator $H \in \mathbb{C}^{d\times d}$ with $H^\dagger = H$.

The random evolution of the state $\rho_k \in \mathcal{D}$ at time step k is modeled through the following Markov process:

$$\rho_{k+1} = \mathbb{U}_{u_k}(\mathbb{M}_{\mu_k}(\rho_k)), \tag{4.1}$$

where

- $u_k \in \mathbb{R}$ is the control at step k,
- μ_k is a random variable taking values μ in $\{1, \ldots, m\}$ with probability $p_{\mu,\rho_k} = \mathrm{Tr}\left(M_\mu \rho_k M_\mu^\dagger\right)$,
- $\mathbb{U}_u$ is the superoperator

$$\mathbb{U}_u : \quad \mathcal{D} \ni \rho \mapsto U_u \rho U_u^\dagger \in \mathcal{D},$$

- for each μ, $\mathbb{M}_\mu$ is the superoperator

$$\mathbb{M}_\mu : \quad \rho \mapsto \frac{M_\mu \rho M_\mu^\dagger}{\mathrm{Tr}\left(M_\mu \rho M_\mu^\dagger\right)} \in \mathcal{D}$$

  defined for any $\rho \in \mathcal{D}$ such that $\mathrm{Tr}\left(M_\mu \rho M_\mu^\dagger\right) \neq 0$.

We suppose throughout this section that the two following assumptions are verified by the system under consideration.

Assumption 4.1. *The measurement operators M_μ are diagonal in the same orthonormal basis $\{\,|n\rangle \in \mathbb{C}^d \mid \quad n \in \{1, \dots, d\}\}$, therefore $M_\mu = \sum_{n=1}^d c_{\mu,n} |n\rangle \langle n|$ with $c_{\mu,n} \in \mathbb{C}$.*

Assumption 4.2. *For all $n_1 \neq n_2$ in $\{1, \dots, d\}$, there exists a $\mu \in \{1, \dots, m\}$ such that $|c_{\mu,n_1}|^2 \neq |c_{\mu,n_2}|^2$.*

Assumption 4.1 means that the considered measurement process achieves a quantum non-demolition (QND) measurement for the physical observables given by orthogonal projections over the states $\left\{\,|n\rangle \in \mathbb{C}^d \mid \quad n \in \{1, \dots, d\}\right\}$. This implies that for $u_k \equiv 0$, any $\rho = |n\rangle \langle n|$ corresponding to the orthogonal projector on the basis vector $|n\rangle$, $n \in \{1, \dots, d\}$, is a fixed point of the Markov process (4.1). Since the operators M_μ must satisfy $\sum_{\mu=1}^m M_\mu^\dagger M_\mu = \mathbb{1}$, we have, according to Assumption 4.1, that $\sum_{\mu=1}^m |c_{\mu,n}|^2 = 1$ for all $n \in \{1, \dots, d\}$.

Assumption 4.2 means that there exists a μ such that the statistics when $u_k \equiv 0$ for obtaining the measurement result μ are different for the fixed points $|n_1\rangle \langle n_1|$ and $|n_2\rangle \langle n_2|$. This follows by observing that $\mathrm{Tr}\left(M_\mu |n\rangle \langle n| M_\mu^\dagger\right) = |c_{\mu,n}|^2$ for $n \in \{1, \dots, d\}$.

4.1.2 Convergence of the open-loop dynamics

When the control vanishes ($u_k = 0, \ \forall k$), the dynamics is simply given by

$$\rho_{k+1} = \mathbb{M}_{\mu_k}(\rho_k), \tag{4.2}$$

where μ_k is a random variable with discrete values in $\{1, \dots, m\}$. The probability p_{μ,ρ_k} to have $\mu_k = \mu$ depends on ρ_k: $p_{\mu,\rho_k} = \mathrm{Tr}\left(M_\mu \rho_k M_\mu^\dagger\right)$. We then have the following theorem which characterizes the open-loop asymptotic behavior.

Theorem 4.1. *Consider a Markov process ρ_k obeying the dynamics of (4.2) with an initial condition ρ_0 in $\mathcal{D}$. Then*

- *with probability one, ρ_k converges to one of the d states $|n\rangle \langle n|$ with $n \in \{1, \dots, d\}$.*
- *the probability of convergence towards the state $|n\rangle \langle n|$ depends only on the initial condition ρ_0 and is given by*

$$\mathrm{Tr}\left(\rho_0 |n\rangle \langle n|\right) = \langle n| \rho_0 |n\rangle .$$

The proof is a generalization of the one given for Theorem 3.1.

Proof. For any $n \in \{1, \dots, d\}$, $\langle n| \rho |n\rangle$ is a martingale. This results from

$$\begin{aligned}\mathbb{E}\left[\langle n| \rho_{k+1} |n\rangle |\rho_k\right] &= \sum_{\mu=1}^{m} \mathrm{Tr}\left(M_\mu \rho_k {M_\mu}^\dagger\right) \langle n| \mathbb{M}_\mu(\rho_k) |n\rangle \\ &= \sum_{\mu=1}^{m} \langle n| M_\mu \rho_k {M_\mu}^\dagger |n\rangle = \langle n| \sum_{\mu=1}^{m} {M_\mu}^\dagger M_\mu \rho_k |n\rangle = \langle n| \rho_k |n\rangle ,\end{aligned}$$

where we have used that M_μ and $|n\rangle \langle n|$ commute, and $\sum_{\mu=1}^{m} {M_\mu}^\dagger M_\mu = \mathbb{1}$.

Take the function

$$V(\rho) := -\sum_{n=1}^{d} \frac{\left(\langle n| \rho |n\rangle \right)^2}{2}. \tag{4.3}$$

The function V being concave and each $\langle n| \rho |n\rangle$ being a martingale, we infer that $V(\rho)$ is a supermartingale, i.e.,

$$\mathbb{E}\left[V(\rho_{k+1})|\rho_k\right] \leq V(\rho_k).$$

More precisely, we have

$$\mathbb{E}\left[V(\rho_{k+1})|\rho_k\right] = -\frac{1}{2} \sum_{n=1}^{d} \sum_{\mu \in I_{\rho_k}} \mathrm{Tr}\left(M_\mu \rho_k {M_\mu}^\dagger\right) \left(\tfrac{\langle n| M_\mu \rho_k {M_\mu}^\dagger |n\rangle}{\mathrm{Tr}\left(M_\mu \rho_k {M_\mu}^\dagger\right)} \right)^2 ,$$

with $I_{\rho_k} = \left\{ \mu \in \{1, \dots, m\} \mid \mathrm{Tr}\left(M_\mu \rho_k {M_\mu}^\dagger\right) \neq 0 \right\}$. We have the identity

$$\begin{aligned}\sum_{\mu \in I_{\rho_k}} \mathrm{Tr}\left(M_\mu \rho_k {M_\mu}^\dagger\right) \left(\tfrac{\langle n| M_\mu \rho_k {M_\mu}^\dagger |n\rangle}{\mathrm{Tr}\left(M_\mu \rho_k {M_\mu}^\dagger\right)} \right)^2 &= \left(\sum_{\mu \in I_{\rho_k}} \mathrm{Tr}\left(M_\mu \rho_k {M_\mu}^\dagger\right) \tfrac{\langle n| M_\mu \rho_k {M_\mu}^\dagger |n\rangle}{\mathrm{Tr}\left(M_\mu \rho_k {M_\mu}^\dagger\right)} \right)^2 \\ &+ \frac{1}{2} \sum_{\mu,\nu \in I_{\rho_k}} \mathrm{Tr}\left(M_\mu \rho_k {M_\mu}^\dagger\right) \mathrm{Tr}\left(M_\nu \rho_k {M_\nu}^\dagger\right) \left(\tfrac{\langle n| M_\mu \rho_k {M_\mu}^\dagger |n\rangle}{\mathrm{Tr}\left(M_\mu \rho_k {M_\mu}^\dagger\right)} - \tfrac{\langle n| M_\nu \rho_k {M_\nu}^\dagger |n\rangle}{\mathrm{Tr}\left(M_\nu \rho_k {M_\nu}^\dagger\right)} \right)^2,\end{aligned}$$

where we have used $\sum_{\mu \in I_{\rho_k}} \mathrm{Tr}\left(M_\mu \rho_k {M_\mu}^\dagger\right) = 1$.

The above identity yields $\mathbb{E}\left[V(\rho_{k+1})|\rho_k\right] - V(\rho_k) = -Q(\rho_k)$ with

$$Q(\rho) = \tfrac{1}{4} \sum_{n=1}^{d} \sum_{\mu,\nu \in I_\rho} \mathrm{Tr}\left(M_\mu \rho {M_\mu}^\dagger\right) \mathrm{Tr}\left(M_\nu \rho {M_\nu}^\dagger\right) \left(\tfrac{|c_{\mu,n}|^2 \langle n|\rho|n\rangle}{\mathrm{Tr}\left(M_\mu \rho {M_\mu}^\dagger\right)} - \tfrac{|c_{\nu,n}|^2 \langle n|\rho|n\rangle}{\mathrm{Tr}\left(M_\nu \rho {M_\nu}^\dagger\right)} \right)^2 .$$

We have used that $\langle n| \mathbb{M}_\mu(\rho) |n\rangle = \frac{|c_{\mu,n}|^2 \langle n|\rho|n\rangle}{\mathrm{Tr}\left(M_\mu \rho {M_\mu}^\dagger\right)}$ and $\sum_{\mu \in I_{\rho_k}} \langle n| M_\mu \rho {M_\mu}^\dagger |n\rangle = \langle n| \rho |n\rangle$. Since $Q \geq 0$, $V(\rho_k)$ is a supermartingale. Note that the sum in the definition of $Q(\rho)$

is over all $\mu, \nu \in I_\rho$. However, we can assume that the sum is actually over all μ, ν in $\{1, \dots, m\}$ by observing the following facts. For any μ, ν, take the mapping

$$\rho \mapsto \mathrm{Tr}\left(M_\mu \rho {M_\mu}^\dagger\right) \mathrm{Tr}\left(M_\nu \rho {M_\nu}^\dagger\right) \left(\tfrac{|c_{\mu,n}|^2 \langle n|\rho|n\rangle}{\mathrm{Tr}\left(M_\mu \rho {M_\mu}^\dagger\right)} - \tfrac{|c_{\nu,n}|^2 \langle n|\rho|n\rangle}{\mathrm{Tr}\left(M_\nu \rho {M_\nu}^\dagger\right)} \right)^2,$$

defined only when $\mathrm{Tr}\left(M_\mu \rho {M_\mu}^\dagger\right) \mathrm{Tr}\left(M_\nu \rho {M_\nu}^\dagger\right) > 0$. Since this mapping is positive and bounded by $\rho \mapsto \mathrm{Tr}\left(M_\mu \rho {M_\mu}^\dagger\right) \mathrm{Tr}\left(M_\nu \rho {M_\nu}^\dagger\right)$, it can be extended by continuity to any $\rho \in \mathcal{D}$ by taking a null value when $\mathrm{Tr}\left(M_\mu \rho {M_\mu}^\dagger\right) \mathrm{Tr}\left(M_\nu \rho {M_\nu}^\dagger\right) = 0$. Thus

$$Q(\rho) = \tfrac{1}{4} \sum_{n=1}^{d} \sum_{\mu,\nu=1}^{m} \mathrm{Tr}\left(M_\mu \rho {M_\mu}^\dagger\right) \mathrm{Tr}\left(M_\nu \rho {M_\nu}^\dagger\right) \left(\tfrac{|c_{\mu,n}|^2 \langle n|\rho|n\rangle}{\mathrm{Tr}\left(M_\mu \rho {M_\mu}^\dagger\right)} - \tfrac{|c_{\nu,n}|^2 \langle n|\rho|n\rangle}{\mathrm{Tr}\left(M_\nu \rho {M_\nu}^\dagger\right)} \right)^2 \tag{4.4}$$

is continuously defined for any $\rho \in \mathcal{D}$ and still satisfies

$$\mathbb{E}\left[V(\rho_{k+1})|\rho_k\right] - V(\rho_k) = -Q(\rho_k), \quad \forall \rho_k \in \mathcal{D}.$$

By Theorem A.4 of the Appendix, the ω-limit set is a subset of $\{\rho \in \mathcal{D} | \quad Q(\rho) = 0\}$. The condition $Q = 0$ implies that, for all n, μ, ν

$$\mathrm{Tr}\left(M_\nu \rho {M_\nu}^\dagger\right) \langle n| M_\mu \rho {M_\mu}^\dagger |n\rangle = \mathrm{Tr}\left(M_\mu \rho {M_\mu}^\dagger\right) \langle n| M_\nu \rho {M_\nu}^\dagger |n\rangle .$$

Taking the sum over all ν, we get for all n and μ,

$$\langle n| M_\mu \rho {M_\mu}^\dagger |n\rangle = \mathrm{Tr}\left(M_\mu \rho {M_\mu}^\dagger\right) \langle n| \rho |n\rangle .$$

These relations read $(\rho_{n',n'} = \langle n'| \rho |n'\rangle)$

$$|c_{\mu,n}|^2 \rho_{n,n} = \left(\sum_{n'} |c_{\mu,n'}|^2 \rho_{n',n'} \right) \rho_{n,n}. \tag{4.5}$$

Since $\mathrm{Tr}\,(\rho) = 1$ and $\rho \geq 0$, there exists $n^* \in \{1, \dots, d\}$ such that $\rho_{n^*,n^*} > 0$. Thus (4.5) with arbitrary μ and $n = n^*$ gives $|c_{\mu,n^*}|^2 = \sum_{n'} |c_{\mu,n'}|^2 \rho_{n',n'}$. Consequently, for all μ and n, (4.5) reads

$$(|c_{\mu,n}|^2 - |c_{\mu,n^*}|^2)\rho_{n,n} = 0.$$

By assumption 4.2, $\rho_{n,n} = 0$ as soon as $n \neq n^*$ and thus $\rho = |n^*\rangle \langle n^*|$. This finishes the proof of the first assertion.

We have shown that the probability measure associated to the random variable ρ_k converges weakly to the probability measure $\sum_{n=1}^{d} p_n \delta_{|n\rangle\langle n|}$, where $\delta_{|n\rangle\langle n|}$ denotes the Dirac measure at $|n\rangle \langle n|$ and p_n is the probability of convergence towards $|n\rangle \langle n|$. In particular, we have $\mathbb{E}\left[\langle n| \rho_k |n\rangle\right] \longrightarrow p_n$. But $\langle n| \rho_k |n\rangle$ is a martingale, hence, $\mathbb{E}\left[\langle n| \rho_k |n\rangle\right] = \mathbb{E}\left[\langle n| \rho_0 |n\rangle\right]$ and consequently, $p_n = \langle n| \rho_0 |n\rangle$. □

4.1.3 Feedback stabilization

Design of strict control-Lyapunov functions. The goal is to design a feedback law that globally stabilizes the Markov chain (4.1) towards a chosen target state $|\bar{n}\rangle\langle\bar{n}|$, for some $\bar{n}$ among $\{1,\ldots,d\}$. In previous publications [71, 68, 36], the proposed feedback schemes tend to decrease at each step the same open-loop martingale $V_{\bar{n}}(\rho) = 1 - \langle\bar{n}|\,\rho\,|\bar{n}\rangle$: u is chosen in order to decrease $u \mapsto V_{\bar{n}}(\mathbb{U}_u(\rho))$. When $\rho = |n\rangle\langle n|$ with $n \neq \bar{n}$, $u \mapsto V_n(\rho)$ is maximum at $u = 0$. Consequently, its first-order u-derivative vanishes at $u = 0$. But its second-order u-derivative could also vanish at $u = 0$. For the photon box considered in the previous chapter, this happens when $|n - \bar{n}| \geq 2$. Such lack of strong concavity in u when the image of ρ is almost orthogonal to $|\bar{n}\rangle$, explains the fact that, in previous works, the control u was set to be a constant non-zero value when $V_{\bar{n}}(\rho)$ is close to one.

To improve convergence and avoid such constant feedback zone, we propose to modify $V_{\bar{n}}$ using the other open-loop martingales $V_n(\rho) = \langle n|\,\rho\,|n\rangle$ and the supermartingale $V(\rho) = -\sum_{n=1}^{d} \frac{(\langle n|\rho|n\rangle)^2}{2}$ used in the proof of Theorem 4.1. The goal of such modification is to get control-Lyapunov functions still admitting a unique global minimum at $|\bar{n}\rangle\langle\bar{n}|$ but being strongly concave versus u around 0 when ρ is close to any $|n\rangle\langle n|$, $n \neq \bar{n}$.

Take $V_0(\rho) = \sum_{n=1}^{d} \sigma_n \langle n|\,\rho\,|n\rangle$ with real coefficients σ_n to be chosen such that $\sigma_{\bar{n}}$ remains the smallest one and such that for any $n \neq \bar{n}$, the second-order u-derivative of $V_0(\mathbb{U}_u(|n\rangle\langle n|))$ at $u = 0$ is strictly negative. This yields to a set of linear equations (see Lemma 4.2) in σ_n that can be solved by inverting a Laplacian matrix (see Lemma 4.1). Notice that V_0 is an open-loop martingale. To obtain a supermartingale we consider (see Theorem 4.2) $V_\epsilon(\rho) = V_0(\rho) + \epsilon V(\rho)$. For $\epsilon > 0$ small enough, $V_\epsilon(\rho)$ still admits a unique global minimum at $|\bar{n}\rangle\langle\bar{n}|$; for u close to zero, $V_\epsilon(\mathbb{U}_u(|n\rangle\langle n|))$ is strongly concave for any $n \neq \bar{n}$ and strongly convex for $n = \bar{n}$. This implies that V_ϵ is a control-Lyapunov function with arbitrary small control (see proof of Theorem 4.2).

In order to formalize our results, we provide next some definitions and lemmas. These will be used later to underlay the construction of the strict control-Lyapunov functions V_ϵ.

Connectivity graph and Laplacian matrix. To the Hamiltonian operator H defining the controlled unitary evolution $U_u = e^{-\imath u H}$, we associate its undirected connectivity graph denoted by G^H. This graph admits d vertices labeled by $n \in \{1,\ldots,d\}$. Two different vertices $n_1 \neq n_2$ ($n_1, n_2 \in \{1,\ldots,d\}$) are linked by an edge if and only if $\langle n_1|\,H\,|n_2\rangle \neq 0$. Attached to H we also associate R^H, the real symmetric matrix $d \times d$ (Laplacian matrix) with entries

$$R^H_{n_1,n_2} = 2\Big(\delta_{n_1,n_2}\langle n_1|\,H^2\,|n_2\rangle - |\langle n_1|\,H\,|n_2\rangle|^2\Big). \tag{4.6}$$

Lemma 4.1. *Assume the graph G^H is connected. Then there exists a vector $\sigma = (\sigma_n)_{n\in\{1,\ldots,d\}}$ of $\mathbb{R}^d$ such that $-R^H\sigma = \lambda$, where λ is a vector of $\mathbb{R}^d$ with components λ_n and $\lambda_{\bar{n}} = -\sum_{n\neq\bar{n}} \lambda_n$.*

Proof. Note that R^H is symmetric and the sum of the entries for any column and any row of R^H is equal to zero. Therefore, the vector $(1\dots 1)^T$ is in the kernel of R^H. The diagonal (resp. non-diagonal) components of R^H are positive (resp. negative). Therefore R^H is a Laplacian matrix (see [14, Ch. 4]). The connectivity graph associated to R^H coincides with G^H. Since this graph is supposed to be connected, classical results of graph theory (see, e.g., [14, Theorem 3.1]) imply that the vector $(1,\dots,1)^T$ spans the kernel of R^H. Therefore, the dimension of the image of R^H is equal to $d-1$. Since R^H is symmetric, its image coincides with the orthogonal to its kernel. For the sake of completeness, here we give a simple proof of this statement. Indeed, for any vector X in the kernel of R^H, we have

$$\sum_{n_1,n_2\in\{1,\dots,d\}} R^H_{n_1,n_2}(X_{n_1}-X_{n_2})^2 = X^T R X = 0.$$

This implies $R^H_{n_1,n_2}(X_{n_1}-X_{n_2})^2=0$, $\forall n_1,n_2\in\{1,\dots,d\}$. As the graph of R^H is connected, we necessarily have $X_{n_1}=X_{n_2}$ for all $n_1,n_2\in\{1,\dots,d\}$. Thus any vector orthogonal to $(1\dots 1)^T$ is in the image of R^H. The vector λ is obviously orthogonal to $(1,\dots,1)^T$. □

Lemma 4.2. *Take $\bar{n}\in\{1,\dots,d\}$ and consider a vector λ in $\mathbb{R}^d$ with components λ_n and $\lambda_{\bar{n}}=-\sum_{n\neq\bar{n}}\lambda_n$. Assume G^H connected and consider a vector $\sigma=(\sigma_n)\in\mathbb{R}^d$ given by Lemma 4.1. For any $\rho\in\mathcal{D}$ we set*

$$V_0(\rho)=\sum_{n=1}^{d}\sigma_n \mathrm{Tr}\left(|n\rangle\langle n|\rho\right)=\sum_{n=1}^{d}\sigma_n\langle n|\rho|n\rangle\,. \tag{4.7}$$

Then for any $n\in\{1,\dots,d\}$ we have

$$\left.\frac{d^2V_0\Big(\mathbb{U}_u(|n\rangle\langle n|)\Big)}{du^2}\right|_{u=0}=\lambda_n.$$

Proof. For any n, set $g_n(u)=V_0(\mathbb{U}_u(|n\rangle\langle n|)=V_0(e^{-\imath uH}|n\rangle\langle n|e^{\imath uH})$. The Baker-Campbell-Hausdorff formula yields up to third order terms in u ($[\cdot,\cdot]$ is the commutator):

$$\mathbb{U}_u(|n\rangle\langle n|)\approx|n\rangle\langle n|-\imath u[H,|n\rangle\langle n|]-\tfrac{u^2}{2}[H,[H,|n\rangle\langle n|]].$$

Consequently, for any $l\in\{1,\dots,d\}$, we have

$$\begin{aligned}\mathrm{Tr}\left(\mathbb{U}_u(|n\rangle\langle n|)\,|l\rangle\langle l|\right)\approx{}&\mathrm{Tr}\Big(|l\rangle\langle l|\Big(|n\rangle\langle n|-\imath u[H,|n\rangle\langle n|]\Big)\Big)\\&-\tfrac{u^2}{2}\mathrm{Tr}\Big(|l\rangle\langle l|\Big([H,[H,|n\rangle\langle n|]]\Big)\Big)\\&=\Big(\delta_{l,n}+\tfrac{u^2}{2}\mathrm{Tr}\left([H,|n\rangle\langle n|][H,|l\rangle\langle l|]\right)\Big),\end{aligned}\tag{4.8}$$

since $\mathrm{Tr}\left(|l\rangle\langle l|[H,|n\rangle\langle n|]\right)=-\mathrm{Tr}\left([|l\rangle\langle l|,|n\rangle\langle n|]H\right)=0$, because $|l\rangle\langle l|$ commutes with $|n\rangle\langle n|$), and since

$$\mathrm{Tr}\left(|l\rangle\langle l|[H,[H,|n\rangle\langle n|]]\right)=-\mathrm{Tr}\left([H,|n\rangle\langle n|][H,|l\rangle\langle l|]\right).$$

Thus, up to third order terms in u, we have

$$g_n(u) = \sum_{l=1}^{d} \sigma_l \left(\delta_{l,n} + \tfrac{u^2}{2} \mathrm{Tr} \left([H, |n\rangle \langle n|][H, |l\rangle \langle l|] \right) \right).$$

Therefore,

$$\left. \frac{\partial^2 V_0\left(\mathbb{U}_u(|n\rangle\langle n|)\right)}{\partial u^2} \right|_{u=0} = \sum_{l=1}^{d} \sigma_l \mathrm{Tr} \left([H, |n\rangle \langle n|][H, |l\rangle \langle l|] \right).$$

It is not difficult to see that $\mathrm{Tr} \left([H, |n\rangle \langle n|][H, |l\rangle \langle l|] \right) = -R^H_{n,l}$. Thus $\left. \frac{\partial^2 V_0\left(\mathbb{U}_u(|n\rangle\langle n|)\right)}{\partial u^2} \right|_{u=0} = -\sum_{l=1}^{d} R^H_{n,l} \sigma_l$. □

The global stabilizing feedback. The main result of this section is expressed in the following theorem.

Theorem 4.2. *Consider the controlled Markov chain of state ρ_k obeying (4.1). Assume that the graph G^H associated to the Hamiltonian H is connected and that the Kraus operators satisfy Assumptions 4.1 and 4.2. Take $\bar{n} \in \{1, \dots, d\}$ and $d-1$ strictly negative real numbers $\lambda_n < 0$ for $n \in \{1, \dots, d\} \setminus \{\bar{n}\}$, and $\lambda_{\bar{n}} = -\sum_{n \neq \bar{n}} \lambda_n$. Consider the component (σ_n) of $\sigma \in \mathbb{R}^d$ defined by Lemma 4.1. Denote by $\rho_{k+\frac{1}{2}} = \mathbb{M}_{\mu_k}(\rho_k)$ the quantum state just after the measurement outcome μ_k at step k. Take $\bar{u} > 0$ and consider the following feedback law*

$$u_k = K(\rho_{k+\frac{1}{2}}) = \underset{u \in [-\bar{u}, \bar{u}]}{\mathit{argmin}} \Big(V_\epsilon\big(\mathbb{U}_u(\rho_{k+\frac{1}{2}})\big)\Big), \tag{4.9}$$

where the control-Lyapunov function $V_\epsilon(\rho)$ is defined by

$$V_\epsilon(\rho) = \sum_{n=1}^{d} \left(\sigma_n \langle n| \rho |n\rangle - \tfrac{\epsilon}{2} \left(\langle n| \rho |n\rangle \right)^2 \right), \tag{4.10}$$

with the parameter $\epsilon > 0$ not too large to ensure that

$$\forall n \in \{1, \dots, d\} \setminus \{\bar{n}\}, \;\; \lambda_n + 2\epsilon \left((\langle n| H |n\rangle)^2 - \langle n| H^2 |n\rangle \right) > 0.$$

Then for any $\rho_0 \in \mathcal{D}$, the closed-loop trajectory ρ_k converges almost surely to the pure state $|\bar{n}\rangle \langle \bar{n}|$.

The proof relies on the fact that $V_\epsilon(\rho)$ is a strict Lyapunov function for the closed-loop system.

Proof. We have

$$\begin{aligned}
&\mathbb{E}\left[V_\epsilon(\rho_{k+1})|\rho_k\right] - V_\epsilon(\rho_k) = \\
&\quad \sum_\mu p_{\mu,\rho_k}\Big(V_\epsilon\big(\mathbb{U}_{K(\mathbb{M}_\mu(\rho_k))}(\mathbb{M}_\mu(\rho_k))\big) - V_\epsilon(\rho_k)\Big) = \\
&\quad \sum_\mu p_{\mu,\rho_k}\Big(\min_{u\in[-\bar{u},\bar{u}]}\Big(V_\epsilon\big(\mathbb{U}_u(\mathbb{M}_\mu(\rho_k))\big)\Big) - V_\epsilon(\rho_k)\Big) = \\
&\quad \sum_\mu p_{\mu,\rho_k}\Big(V_\epsilon\big(\mathbb{M}_\mu(\rho_k)\big) - V_\epsilon(\rho_k)\Big) + \\
&\quad \sum_\mu p_{\mu,\rho_k}\Big(\min_{u\in[-\bar{u},\bar{u}]}\Big(V_\epsilon\big(\mathbb{U}_u(\mathbb{M}_\mu(\rho_k))\big)\Big) - V_\epsilon\big(\mathbb{M}_\mu(\rho_k)\big)\Big) =: \\
&-\widetilde{Q}_1(\rho_k) - \widetilde{Q}_2(\rho_k),
\end{aligned}$$

with $\widetilde{Q}_1$ and $\widetilde{Q}_2$ given by the following

$$\widetilde{Q}_1(\rho_k) := \sum_\mu p_{\mu,\rho_k}\Big(V_\epsilon(\rho_k) - V_\epsilon\big(\mathbb{M}_\mu(\rho_k)\big)\Big),$$

and

$$\widetilde{Q}_2(\rho_k) := \sum_\mu p_{\mu,\rho_k}\Big(V_\epsilon\big(\mathbb{M}_\mu(\rho_k)\big) - \min_{u\in[-\bar{u},\bar{u}]}\Big(V_\epsilon\big(\mathbb{U}_u(\mathbb{M}_\mu(\rho_k))\big)\Big)\Big).$$

These functions are both positive continuous functions of ρ_k (the continuity of these functions can be proved in the same way as the proof of the continuity of $Q(\rho_k)$ in Theorem 4.1). By Theorem A.4 of the appendix, the ω-limit set Ω is included in the following set

$$\{\rho \in \mathcal{D} |\ \widetilde{Q}_1(\rho) = 0\} \ \cap\ \{\rho \in \mathcal{D} |\ \widetilde{Q}_2(\rho) = 0\}.$$

Indeed $\widetilde{Q}_1$ coincides with Q defined in (4.4). During the proof of Theorem 4.1, we have shown that $\widetilde{Q}_1(\rho) = 0$ implies $\rho = |n\rangle\langle n|$ for some $n \in \{1, \dots, d\}$. But $\widetilde{Q}_2(|n\rangle\langle n|) = 0$ implies that $\min_{u\in[-\bar{u},\bar{u}]} V_\epsilon\big(\mathbb{U}_u(|n\rangle\langle n|)\big) = V_\epsilon\big(\langle n|\,|n\rangle\big)$, since $\mathbb{M}_\mu(|n\rangle\langle n|) = |n\rangle\langle n|$. According to Lemma 4.2 and the relation (4.8), we have $\left.\frac{dV_\epsilon\left(\mathbb{U}_u(|n\rangle\langle n|)\right)}{du}\right|_{u=0} = 0$ and

$$\left.\frac{d^2V_\epsilon\left(\mathbb{U}_u(|n\rangle\langle n|)\right)}{du^2}\right|_{u=0} = \lambda_n + 2\epsilon\left((\langle n| H |n\rangle)^2 - \langle n| H^2 |n\rangle\right).$$

Since $\lambda_n < 0$ for $n \neq \bar{n}$ and ϵ is not too large, for any $n \neq \bar{n}$, $u = 0$ is a locally strict maximum of $V_\epsilon\big(\mathbb{U}_u(|n\rangle\langle n|)\big)$. Consequently, $\widetilde{Q}_2(|n\rangle\langle n|) = 0$ implies that $n = \bar{n}$. □

Remark 4. *In fact, the unitary propagator U_u does not admit in general a simple analytic form. For the case of photon box (see [5, Section 5]), the simulations show that we* *can replace U_u by its quadratic approximation valid for u small and derived from Baker-Campbell-Hausdorff formula:*

$$\mathbb{U}_u(\rho) \approx -iu[H,\rho] - \tfrac{u^2}{2}[H,[H,\rho]] + \mathcal{O}(u^3),$$

with Hamiltonian $H = i(\mathbf{a}^\dagger - \mathbf{a})$. This yields an explicit quadratic approximation of $V_\epsilon(\mathbb{U}_u(\rho))$ around zero. The feedback is then given by minimizing a parabolic expression in u of $V_\epsilon(\mathbb{U}_u(\rho))$. This approximated function takes its minimum necessary at one of the two extremes $-\bar{u}$, $\bar{u}$ or at the point where its derivative is zero.

4.1.4 Quantum filtering and separation principle

When the measurement process is perfect, the Markov process (4.1) represents a natural choice for estimating the hidden state ρ. Thus, the estimate ρ^e of ρ satisfies the following dynamics

$$\rho^e_{k+1} = \mathbb{U}_{u_k}(\mathbb{M}_{\mu_k}(\rho^e_k)), \tag{4.11}$$

where the measurement outcome μ_k is driven by (4.1). We now announce the following theorem which is the analogue of Theorem 3.3.

Theorem 4.3. *Consider the Markov chain of state ρ_k obeying (4.1) and assume that the assumptions of Theorem 4.2 are satisfied. For each measurement outcome μ_k given by (4.1), consider the estimation ρ^e_k given by (4.11) with an initial condition ρ^e_0. Set $u_k = K(\rho^e_{k+\frac{1}{2}})$ where K is given by (4.9). Then for all $\epsilon \in \left]0, \min_{n\neq\bar{n}}\left(\frac{\lambda_n}{R^H_{n,n}}\right)\right]$, we have ρ_k and ρ^e_k converge almost surely towards the target state $|\bar{n}\rangle\langle\bar{n}|$ as soon as $\ker(\rho^e_0) \subset \ker(\rho_0)$.*

The proof of this theorem is similar to the one given for Theorem 3.3, and is omitted.

4.2 Strict control-Lyapunov functions for non-separable Markov models

We generalize here the Lyapunov design proposed in Section 4.1 for arbitrary controlled Kraus operators M^u_μ that cannot be decomposed into QND measurement operators M^0_μ followed by a unique controlled unitary operator U_u with $U_0 = \mathbb{1}$. In [5, 91], it is assumed that $M^u_\mu \equiv U_u M^0_\mu$.

4.2.1 The non-separable Markov model

We consider a quantum system defined in the Hilbert space $\mathcal{H} = \mathbb{C}^d$, and measured through a generalized measurement procedure at discrete-time intervals. To each measurement outcome $\mu \in \{1,\dots,m\}$ is attached the Kraus operator $M^u_\mu \in \mathbb{C}^{d\times d}$ depending on μ and also on a scalar control input $u \in \mathbb{R}$. For each u, these Kraus operators $(M^u_\mu)_{\mu\in\{1,\dots,m\}}$

satisfy the constraint $\sum_{\mu=1}^{m} M_\mu^{u\dagger} M_\mu^u = \mathbb{1}$. They are attached to the Kraus map $\mathbb{K}^u$ defined by

$$\mathcal{D} \ni \rho \mapsto \mathbb{K}^u(\rho) = \sum_{\mu=1}^{m} M_\mu^u \rho M_\mu^{u\dagger} \in \mathcal{D}. \tag{4.12}$$

The random evolution of the state $\rho_k \in \mathcal{D}$ at time step k is modeled through the following Markov process

$$\rho_{k+1} = \mathbb{M}_{\mu_k}^{u_k}(\rho_k), \tag{4.13}$$

where

- u_k is the control at step k,
- μ_k is a random variable taking values μ in $\{1, \dots, m\}$ with probability

$$p_{\mu,\rho_k}^{u_k} = \mathrm{Tr}\left(M_\mu^{u_k} \rho_k M_\mu^{u_k \dagger}\right),$$

- for each μ, $\mathbb{M}_\mu^u$ is the superoperator

$$\mathbb{M}_\mu^u : \quad \rho \mapsto \frac{M_\mu^u \rho M_\mu^{u\dagger}}{\mathrm{Tr}\left(M_\mu^u \rho M_\mu^{u}{}^\dagger\right)} \in \mathcal{D}, \tag{4.14}$$

  defined for $\rho \in \mathcal{D}$ such that $p_{\mu,\rho}^u = \mathrm{Tr}\left(M_\mu^u \rho M_\mu^{u\dagger}\right) \neq 0$.

We state below three essential assumptions that we need in the following. Assumptions 4.3 and 4.4 are the same as Assumptions 4.1 and 4.2 where we replace M_μ by M_μ^0.

Assumption 4.3. *For $u = 0$, the measurement operators M_μ^0 are diagonal in the same orthonormal basis $\{\, |n\rangle \in \mathbb{C}^d \,|\, n \in \{1, \dots, d\}\}$, therefore $M_\mu^0 = \sum_{n=1}^{d} c_{\mu,n} |n\rangle \langle n|$ with $c_{\mu,n} \in \mathbb{C}$.*

Assumption 4.4. *For all $n_1 \neq n_2$ in $\{1, \dots, d\}$, there exists $\mu \in \{1, \dots, m\}$ such that $|c_{\mu,n_1}|^2 \neq |c_{\mu,n_2}|^2$.*

Assumption 4.5. *The measurement operators M_μ^u are C^2 functions of u.*

4.2.2 Convergence of the open-loop dynamics

When the control input vanishes ($u \equiv 0$), the dynamics are simply given by

$$\rho_{k+1} = \mathbb{M}_{\mu_k}^0(\rho_k), \tag{4.15}$$

where μ_k is a random variable with values in $\{1, \dots, m\}$. The probability p_{μ,ρ_k}^0 to have $\mu_k = \mu$ depends on ρ_k: $p_{\mu,\rho_k}^0 = \mathrm{Tr}\left(M_\mu^0 \rho_k M_\mu^{0\dagger}\right)$. The open-loop asymptotic behavior is characterized by the following theorem, whose proof is analogue to the proof of Theorem 4.1 and is so omitted.

Theorem 4.4. *Consider a Markov process ρ_k obeying the dynamics of (4.15) with an initial condition ρ_0 in $\mathcal{D}$. Then*

- *with probability one, ρ_k converges to one of the d states $|n\rangle\langle n|$ with $n \in \{1, \ldots, d\}$,*
- *the probability of convergence towards the state $|n\rangle\langle n|$ depends only on the initial condition ρ_0 and is given by $\langle n|\rho_0|n\rangle$.*

4.2.3 Feedback stabilization

Design of strict control-Lyapunov function. Theorem 4.4 implies that the open-loop dynamics is stable in the sense that almost all realizations ρ_k converges to one of the pure states $|n\rangle\langle n|$ where n is a random variable in $\{1, \ldots, d\}$, taking the value $\bar{n}$ with probability $\langle \bar{n}|\rho_0|\bar{n}\rangle$ (since $\langle \bar{n}|\rho|\bar{n}\rangle$ is a martingale). The control goal is to ensure a deterministic convergence towards a controller set point $\bar{n} \in \{1, \ldots, d\}$. The control design is based on Lyapunov techniques exploiting the d open-loop martingales $\langle n|\rho|n\rangle$. In Chapter 3, we take

$$1 - \langle \bar{n}|\rho|\bar{n}\rangle = \sum_{n \neq \bar{n}} \langle n|\rho|n\rangle$$

as control-Lyapunov function and propose a state feedback law transforming this open-loop martingale into a closed-loop supermartingale. In Section 4.1, the feedback relies on Lyapunov functions $V_0(\rho)$ of the form

$$V_0(\rho) = \sum_{n=1}^{d} \sigma_n \langle n|\rho|n\rangle,$$

where the real coefficients σ_n are chosen such that $\sigma_{\bar{n}}$ remains the smallest one. Since the coefficients σ_n are all different in general, such Lyapunov functions achieve a better discrimination between the open-loop equilibrium points $|n\rangle\langle n|$: a large σ_n indicates that the control effort needed to steer $|n\rangle\langle n|$ towards $|\bar{n}\rangle\langle \bar{n}|$ will be important.

More precisely, these coefficients have to ensure that for any $n \neq \bar{n}$, the second-order u-derivative of $V_0(\mathbb{K}^u(|n\rangle\langle n|))$ at $u = 0$ is strictly negative ($\mathbb{K}^u$ is the Kraus map defined by (4.12)). This yields to a set of linear equations in σ_n that can be solved by inverting an irreducible Metzler matrix (see Lemma 4.4). Once the σ_n's satisfy these constraints, the stabilizing feedback law proposed in Theorem 4.5 is based on the open-loop supermartingale

$$V_\epsilon(\rho) = V_0(\rho) - \tfrac{\epsilon}{2} \sum_{n=1}^{d} (\langle n|\rho|n\rangle)^2.$$

For $\epsilon > 0$ small enough, $V_\epsilon(\rho)$ still admits a unique global minimum at $\rho = |\bar{n}\rangle\langle \bar{n}|$, for u close to 0, $u \mapsto V_\epsilon(\mathbb{K}^u(|n\rangle\langle n|))$ is strongly concave for any $n \neq \bar{n}$ and strongly convex for $n = \bar{n}$. This indicates that V_ϵ is still a good candidate as control-Lyapunov function. The construction of $V_0(\rho)$ relies on the following lemmas.

Lemma 4.3. *Consider the $d \times d$ matrix R defined by*

$$R_{n_1,n_2} = \sum_{\mu} \left(2 \left| \langle n_1 | \tfrac{dM_\mu^{u\dagger}}{du}\big|_{u=0} | n_2 \rangle \right|^2 + 2\delta_{n_1,n_2} \Re\big(c_{\mu,n_1} \langle n_1 | \tfrac{d^2 M_\mu^{u\dagger}}{du^2}\big|_{u=0} | n_2 \rangle \big) \right).$$

When $R \neq 0$, the non-negative $P = \mathbb{1} - R/\mathrm{Tr}\,(R)$ is a right stochastic matrix. The notation $\Re(A)$ denotes the real part of A.

Proof. For $n_1 \neq n_2$, $R_{n_1,n_2} \geq 0$. Thus R is a Metzler matrix. Let us prove that the sum of each row vanishes. This results from the identity $\sum_\mu M_\mu^{u\dagger} M_\mu^u = \mathbb{1}$. Derivating twice with respect to u the relation

$$1 = \sum_\mu \langle n_1 | M_\mu^{u\dagger} M_\mu^u | n_1 \rangle = \sum_{\mu,n_2} \langle n_1 | M_\mu^{u\dagger} | n_2 \rangle \langle n_2 | M_\mu^u | n_1 \rangle ,$$

yields

$$\sum_{\mu,n_2} 2 \langle n_1 | \tfrac{dM_\mu^{u\dagger}}{du} | n_2 \rangle \langle n_2 | \tfrac{dM_\mu^u}{du} | n_1 \rangle + \langle n_1 | \tfrac{d^2 M_\mu^{u\dagger}}{du^2} | n_2 \rangle \langle n_2 | M_\mu^u | n_1 \rangle + \langle n_1 | M_\mu^{u\dagger} | n_2 \rangle \langle n_2 | \tfrac{d^2 M_\mu^u}{du^2} | n_1 \rangle = 0.$$

Since for $u = 0$, $\langle n_2 | M_\mu^0 | n_1 \rangle = \delta_{n_1,n_2} c_{\mu,n_1}$, the above sum corresponds to $\sum_{n_2} R_{n_1,n_2}$. Therefore, the diagonal elements of R are non-positive. If $R \neq 0$, then $\mathrm{Tr}\,(R) < 0$ and the matrix $P = \mathbb{1} - R/\mathrm{Tr}\,(R)$ is well-defined with non-negative entries. Since the sum of each row of R vanished, the sum of each row of P is equal to 1. Thus P is a right stochastic matrix. □

To the Metzler matrix R defined in Lemma 4.3, we associate its directed graph denoted by G. This graph admits d vertices labeled by $n \in \{1, \dots, d\}$. To each strictly positive off-diagonal element of the matrix R, say, on the n_1th row and the n_2th column we associate an arc from vertex n_1 towards vertex n_2.

Lemma 4.4. *Assume the directed graph G is strongly connected, i.e., for any $n, n' \in \{1, \dots, d\}$, $n \neq n'$, there exists a chain of r distinct elements $(n_j)_{j=1,\dots,r}$ of $\{1, \dots, d\}$ such that $n_1 = n$, $n_r = n'$ and for any $j = 1, \dots r-1$, $R_{n_j,n_{j+1}} \neq 0$. Take $\bar{n} \in \{1, \dots, d\}$. Then, there exist $d-1$ strictly positive real numbers e_n, $n \in \{1, \dots, d\} \setminus \{\bar{n}\}$, such that*

- *for any reals λ_n, $n \in \{1, \dots, d\} \setminus \{\bar{n}\}$, there exists a unique vector $\sigma = (\sigma_n)_{n \in \{1,\dots,d\}}$ of $\mathbb{R}^d$ with $\sigma_{\bar{n}} = 0$ such that $R\sigma = \lambda$, where λ is the vector of $\mathbb{R}^d$ of components λ_n for $n \in \{1, \dots, d\} \setminus \{\bar{n}\}$ and $\lambda_{\bar{n}} = -\sum_{n \neq \bar{n}} e_n \lambda_n$. If additionally $\lambda_n < 0$ for all $n \in \{1, \dots, d\} \setminus \{\bar{n}\}$, then $\sigma_n > 0$ for all $n \in \{1, \dots, d\} \setminus \{\bar{n}\}$;*
- *for any vector $\sigma \in \mathbb{R}^d$, solution of $R\sigma = \lambda \in \mathbb{R}^d$, the function $V_0(\rho) = \sum_{n=1}^d \sigma_n \langle n | \rho | n \rangle$ satisfies*

$$\left. \frac{d^2 V_0\big(\mathbb{K}^u(|n\rangle \langle n|)\big)}{du^2} \right|_{u=0} = \lambda_n \qquad \forall n \in \{1, \dots, d\}.$$

Proof. Since the directed graph G coincides with the directed graph of the right stochastic matrix P defined in Lemma 4.3, P is irreducible. Since it is a right stochastic matrix, its spectral radius is equal to 1. By Perron-Frobenius theorem for non-negative irreducible matrices, this spectral radius, i.e., one, is also an eigenvalue of P and of P^T, with multiplicity one and associated to eigenvectors having strictly positive entries. The right eigenvector ($Pw = w$) is obviously $w = (1, \ldots, 1)^T$, and the left eigenvector $e = (e_1, \ldots, e_n)^T$ ($P^T e = e$) can be chosen such that $e_{\bar{n}} = 1$. Consequently, the rank of R is $d-1$ with $\ker(R) = \mathrm{span}(w)$ and $\mathrm{Im}(R) = e^{\perp}$, where $e^{\perp}$ is the hyperplane orthogonal to e. Since $e^T\lambda = 0$, $\lambda \in \mathrm{Im}(R)$, there exists σ such that $R\sigma = \lambda$. Since $\ker(R) = \mathrm{span}(w)$, there is a unique σ solution of $R\sigma = \lambda$ such that $\sigma_{\bar{n}} = 0$. The fact that $\sigma_n > \sigma_{\bar{n}}$ when $\lambda_n < 0$ for $n \neq \bar{n}$, comes from elementary manipulations of $P\sigma = \sigma - \lambda/\mathrm{Tr}\,(R)$ showing that $\min_{n\neq\bar{n}} \sigma_n > \sigma_{\bar{n}}$.

For any n, set $g_n^u = W_0(\mathbb{K}^u(|n\rangle\langle n|)) = \sum_l \sigma_l \langle l|\,\mathbb{K}^u(|n\rangle\langle n|)\,|l\rangle$, and $P_l^u = \langle l|\,\mathbb{K}^u(|n\rangle\langle n|)\,|l\rangle$. We have

$$\tfrac{dP_l^u}{du} = \sum_\mu \left(\langle l|\, \tfrac{dM_\mu^u}{du}\,|n\rangle\langle n|\, M_\mu^{u\dagger}\,|l\rangle + \langle l|\, M_\mu^u\,|n\rangle\langle n|\,\tfrac{dM_\mu^{u\dagger}}{du}\,|l\rangle \right)$$

and

$$\begin{aligned}
\tfrac{d^2P_l^u}{du^2}\big|_{u=0} =& \sum_\mu \Big(\langle l|\, \tfrac{d^2M_\mu^u}{du^2}\big|_{u=0}\,|n\rangle\langle n|\, M_\mu^{0\dagger}\,|l\rangle \\
&+ \langle l|\,\tfrac{dM_\mu^u}{du}\big|_{u=0}\,|n\rangle\langle n|\,\tfrac{dM_\mu^{u\dagger}}{du}\big|_{u=0}\,|l\rangle + \langle l|\, M_\mu^0\,|n\rangle\langle n|\,\tfrac{d^2M_\mu^{u\dagger}}{du^2}\big|_{u=0}\,|l\rangle \Big) \\
=& \sum_\mu \left(2\left|\langle n|\,\tfrac{dM_\mu^{u\dagger}}{du}\big|_{u=0}\,|l\rangle\right|^2 + 2\delta_{nl}\Re\Big(c_{\mu,n}\,\langle n|\,\tfrac{d^2M_\mu^{u\dagger}}{du^2}\big|_{u=0}\,|l\rangle\Big)\right).
\end{aligned}$$

Therefore $\frac{d^2P_l^u}{du^2}|_{u=0} = R_{n,l}$ and $\left.\frac{d^2W_0\left(\mathbb{K}^u(|n\rangle\langle n|)\right)}{du^2}\right|_{u=0} = \sum_{l=1}^d R_{n,l}\sigma_l = \lambda_n$. ☐

The global stabilizing feedback. We now state the main result of this section.

Theorem 4.5. *Consider the Markov chain (4.13) with Assumptions 4.3, 4.4 and 4.5. Take $\bar{n} \in \{1, \ldots, d\}$ and assume that the directed graph G associated to the Metzler matrix R of Lemma 4.3 is strongly connected. Take $\epsilon > 0$, $\sigma \in \mathbb{R}_+^d$ the solution of $R\sigma = \lambda$ with $\sigma_{\bar{n}} = 0$, $\lambda_n < 0$ for $n \in \{1, \ldots, d\} \setminus \{\bar{n}\}$, $\lambda_{\bar{n}} = -\sum_{n\neq\bar{n}} e_n\lambda_n$ (see Lemma 4.4) and consider*

$$V_\epsilon(\rho) = \sum_{n=1}^d \sigma_n \langle n|\,\rho\,|n\rangle - \tfrac{\epsilon}{2}(\langle n|\,\rho\,|n\rangle)^2.$$

Take $\bar{u} > 0$ and consider the following feedback law

$$u_k = \underset{u\in[-\bar{u},\bar{u}]}{\mathit{argmin}}\Big(\mathbb{E}\left[V_\epsilon(\rho_{k+1})|\rho_k, u_k = u\right]\Big) =: f(\rho_k). \tag{4.16}$$

Then for all $\epsilon \in \left]0, \min_{n\neq\bar{n}}\left(\frac{\lambda_n}{R_{n,n}}\right)\right]$, the closed-loop Markov chain of state ρ_k with the feedback law (4.16) converges almost surely towards $|\bar{n}\rangle\langle\bar{n}|$ for any initial condition $\rho_0 \in \mathcal{D}$.

The proof of this theorem is very similar to the one given for Theorem 4.2.

Proof. We have

$$\begin{aligned}
&\mathbb{E}\left[V_\epsilon(\rho_{k+1})|\rho_k\right] - V_\epsilon(\rho_k) = \\
&\sum_\mu p^{f(\rho_k)}_{\mu,\rho_k} V_\epsilon(\mathbb{M}^{f(\rho_k)}_\mu(\rho_k)) - V_\epsilon(\rho_k) = \\
&\min_{u\in[-\bar{u},\bar{u}]} \sum_\mu p^u_{\mu,\rho_k} V_\epsilon(\mathbb{M}^u_\mu(\rho_k)) - V_\epsilon(\rho_k) = \\
&\min_{u\in[-\bar{u},\bar{u}]} \sum_\mu p^u_{\mu,\rho_k} V_\epsilon(\mathbb{M}^u_\mu(\rho_k)) - \sum_\mu p^0_{\mu,\rho_k} V_\epsilon(\mathbb{M}^0_\mu(\rho_k)) + \sum_\mu p^0_{\mu,\rho_k} V_\epsilon(\mathbb{M}^0_\mu(\rho_k)) - V_\epsilon(\rho_k) := \\
&-\mathcal{Q}_1 - \mathcal{Q}_2,
\end{aligned}$$

with $\mathcal{Q}_1$ and $\mathcal{Q}_2$ defined by the following:

$$\mathcal{Q}_1 := V_\epsilon(\rho_k) - \sum_\mu p^0_{\mu,\rho_k} V_\epsilon(\mathbb{M}^0_\mu(\rho_k)),$$

and

$$\mathcal{Q}_2 := \sum_\mu p^0_{\mu,\rho_k} V_\epsilon(\mathbb{M}^0_\mu(\rho_k)) - \min_{u\in[-\bar{u},\bar{u}]} \sum_\mu p^u_{\mu,\rho_k} V_\epsilon(\mathbb{M}^u_\mu(\rho_k)).$$

These functions are both positive continuous function of ρ_k . By Theorem A.4 of the appendix, the ω-limit set Ω is included in the following set

$$\{\rho \in \mathcal{D}|\ \mathcal{Q}_1(\rho) = 0\} \ \cap\ \{\rho \in \mathcal{D}|\ \mathcal{Q}_2(\rho) = 0\}.$$

Indeed $\mathcal{Q}_1$ coincides with Q defined in (4.4), if we replace M_μ by M^0_μ, so $\mathcal{Q}_1(\rho) = 0$ implies $\rho = |n\rangle\langle n|$ for some $n \in \{1, \ldots, d\}$. But $\mathcal{Q}_2(|n\rangle\langle n|) = 0$ implies that

$$\min_{u\in[-\bar{u},\bar{u}]} \sum_\mu p^u_{\mu,\rho_k} V_\epsilon(\mathbb{M}^u_\mu(|n\rangle\langle n|)) = V_\epsilon(|n\rangle\langle n|),$$

since $\mathbb{M}^0_\mu(|n\rangle\langle n|) = |n\rangle\langle n|$ and $\sum_\mu p^0_{\mu,|n\rangle\langle n|} = 1$. According to Lemma 4.4, we have

$$\frac{dV_\epsilon(\mathbb{K}^u(|n\rangle\langle n|))}{du}\Big|_{u=0} = 0 \quad \text{and} \quad \frac{d^2V_\epsilon(\mathbb{K}^u(|n\rangle\langle n|))}{du^2}\Big|_{u=0} = \lambda_n + \epsilon R_{nn}.$$

Now remark that $\lambda_n < 0$ for $n \neq \bar{n}$ and ϵ is not too large, then for any $n \neq \bar{n}$, $u = 0$ is a locally strict maximum of $V_\epsilon(\mathbb{M}^u_\mu(|n\rangle\langle n|))$. As a result, $\mathcal{Q}_2(|n\rangle\langle n|) = 0$ implies that $n = \bar{n}$. □

4.2.4 Quantum filtering and separation principle

The natural estimate ρ^e of ρ satisfies the following dynamics,

$$\rho^e_{k+1} = \mathbb{M}^{u_k}_{\mu_k}(\rho^e_k). \tag{4.17}$$

We now state the following theorem which ensures the convergence of ρ_k and ρ^e_k towards the target state $\bar{\rho} = |\bar{n}\rangle\langle\bar{n}|$.

Theorem 4.6. *Consider the Markov chain of state ρ_k obeying (4.13) and assume that the assumptions of Theorem 4.5 are satisfied. For each measurement outcome μ_k given by (4.13), consider the estimation ρ^e_k given by (4.17) with an initial condition ρ^e_0. Set $u_k = f(\rho^e_k)$ where f is given by (4.16). Then for all $\epsilon \in \left]0, \min_{n\neq\bar{n}}\left(\frac{\lambda_n}{R_{n,n}}\right)\right]$, ρ_k and ρ^e_k converge almost surely towards the target state $|\bar{n}\rangle\langle\bar{n}|$ as soon as $\ker(\rho^e_0) \subset \ker(\rho_0)$.*

Proof. We demonstrate that $\mathbb{E}\left[\mathrm{Tr}\left(\rho_k\bar{\rho}\right)|\rho_0,\rho^e_0\right]$ (where we take the expectation over all jump realizations) depends linearly on ρ_0 even though we apply the feedback control.

Since u_k is a function of ρ^e_0, we have

$$\mathbb{E}\left[\mathrm{Tr}\left(\rho_k\bar{\rho}\right)|\rho_0,\rho^e_0\right] = \sum_{\mu_0,\ldots,\mu_{k-1}} \mathrm{Tr}\left(\widetilde{M}^{u_k}_{\mu_{k-1}}\left(\ldots\left(\widetilde{M}^{u_0}_{\mu_0}(\rho_0)\right)\right)\bar{\rho}\right),$$

where $\widetilde{M}^u_\mu(\rho) = M^u_\mu \rho M^{u\dagger}_\mu$. The linearity of $\mathbb{E}\left[\mathrm{Tr}\left(\rho_k\bar{\rho}\right)|\rho_0,\rho^e_0\right]$ with respect to ρ_0 is thus verified. The rest of the proof relies on the same arguments presented for Theorem 3.3. □

4.3 Compensation of feedback delays

In this section, we take delays into account.

4.3.1 The non-separable Markov model

The random evolution of the state $\rho_k \in \mathcal{D}$ at time step k is modeled through the following Markov process

$$\rho_{k+1} = \mathbb{M}^{u_{k-\tau}}_{\mu_k}(\rho_k), \tag{4.18}$$

where

- $u_{k-\tau}$ is the control at step k, subject to a delay of $\tau > 0$ steps. This delay is usually due to delays in the measurement process that can also be seen as delays in the control process;
- μ_k is a random variable taking values μ in $\{1,\ldots,m\}$ with probability
$$p^{u_{k-\tau}}_{\mu,\rho_k} = \mathrm{Tr}\left(M^{u_{k-\tau}}_\mu \rho_k M^{u_{k-\tau}\dagger}_\mu\right);$$
- for each μ, $\mathbb{M}^u_\mu$ is the superoperator defined in (4.14).

We suppose that the system under consideration verifies Assumptions 4.3, 4.4 and 4.5. Thus, it is clear that the open-loop behavior is described through Theorem 4.4.

4.3.2 Feedback stabilization

Design of the control-Lyapunov function. We take the same control-Lyapunov function V_ϵ which is proposed in Section 4.2. Now it remains to explain how to take into account the delay τ explicitly into the Lyapunov design relying of $V_\epsilon(\rho)$. Take $(\rho_k, u_{k-1}, \ldots, u_{k-\tau})$ as state at step k. More precisely denote by $\chi = (\rho, \beta_1, \ldots, \beta_\tau)$ this state where β_l stands for the control input u delayed l steps. Then the state form of the delay dynamics (4.18) is governed by the following Markov chain

$$\begin{cases} \rho_{k+1} &= \mathbb{M}_{\mu_k}^{\beta_{\tau,k}}(\rho_k) \\ \beta_{1,k+1} &= u_k \\ \beta_{2,k+1} &= \beta_{1,k} \\ & \vdots \\ \beta_{\tau,k+1} &= \beta_{\tau-1,k}. \end{cases} \tag{4.19}$$

The goal is to design a feedback law $u_k = \mho(\chi_k)$ that globally stabilizes this Markov chain towards a chosen target state $\bar{\chi} = (|\bar{n}\rangle\langle\bar{n}|, 0, \ldots, 0)$ for some $\bar{n} \in \{1, \ldots, d\}$. In Theorem 4.7, we show how to design a feedback relying on the control-Lyapunov function

$$W_\epsilon(\chi) = V_\epsilon(\mathbb{K}^{\beta_1}(\mathbb{K}^{\beta_2}(\ldots\ldots\mathbb{K}^{\beta_\tau}(\rho)\ldots))),$$

where $V_\epsilon(\rho) = V_0(\rho) + \epsilon V(\rho)$. The construction of $V_0(\rho)$ relies on Lemmas 4.3 and 4.4.

The global stabilizing feedback. The main result of this section is the following theorem.

Theorem 4.7. *Consider the Markov chain (4.19) with Assumptions 4.3, 4.4 and 4.5. Take $\bar{n} \in \{1, \ldots, d\}$ and assume that the directed graph G associated to the Metzler matrix R of Lemma 4.3 is strongly connected. Take $\epsilon > 0$, and let $\sigma \in \mathbb{R}_+^d$ be the solution of $R\sigma = \lambda$ with $\sigma_{\bar{n}} = 0$, $\lambda_n < 0$ for $n \in \{1, \ldots, d\} \setminus \{\bar{n}\}$, $\lambda_{\bar{n}} = -\sum_{n \neq \bar{n}} e_n \lambda_n$ (see Lemma 4.4) and consider*

$$V_\epsilon(\rho) = \sum_{n=1}^{d} \sigma_n \langle n|\rho|n\rangle - \tfrac{\epsilon}{2}(\langle n|\rho|n\rangle)^2.$$

Take $\bar{u} > 0$ and consider the following feedback law

$$u_k = \underset{\xi \in [-\bar{u}, \bar{u}]}{\mathit{argmin}} \Big(\mathbb{E}\left[W_\epsilon(\chi_{k+1}) | \chi_k, u_k = \xi\right]\Big) =: \mho(\chi_k), \tag{4.20}$$

where $W_\epsilon(\chi) = V_\epsilon(\mathbb{K}^{\beta_1}(\mathbb{K}^{\beta_2}(\ldots\ldots\mathbb{K}^{\beta_\tau}(\rho)\ldots)))$.
Then, there exists $u^* > 0$ such that for all $\bar{u} \in]0, u^*]$ and $\epsilon \in \left]0, \min_{n \neq \bar{n}}\left(\frac{\lambda_n}{R_{n,n}}\right)\right]$, the closed-loop Markov chain of state χ_k with the feedback law (4.20) converges almost surely towards $(|\bar{n}\rangle\langle\bar{n}|, 0, \ldots, 0)$ for any initial condition $\chi_0 \in \mathcal{D} \times [-\bar{u}, \bar{u}]^\tau$.

Proof. For the sake of simplicity, first we demonstrate this theorem for $\tau = 1$ and thus for $\chi = (\rho, \beta_1)$. We then explain how this proof may be extended to arbitrary $\tau > 1$.

We have

$$\mathbb{E}\left[W_\epsilon(\chi_{k+1})|\chi_k, u_k\right] = \sum_\mu p_{\mu,\rho_k}^{\beta_{1,k}} V_\epsilon(\mathbb{K}^{u_k}(\mathbb{M}_\mu^{\beta_{1,k}}(\rho_k))).$$

Thus $\mathbb{E}\left[W_\epsilon(\chi_{k+1})|\chi_k, u_k\right] = -Q_1(\chi_k) + \sum_\mu p_{\mu,\rho_k}^{\beta_{1,k}} V_\epsilon(\mathbb{K}^0(\mathbb{M}_\mu^{\beta_{1,k}}(\rho_k)))$, where we have set

$$Q_1(\chi_k) = \sum_\mu p_{\mu,\rho_k}^{\beta_{1,k}} V_\epsilon(\mathbb{K}^0(\mathbb{M}_\mu^{\beta_{1,k}}(\rho_k))) - \min_{\xi\in[-\bar{u},\bar{u}]} \sum_\mu p_{\mu,\rho_k}^{\beta_{1,k}} V_\epsilon(\mathbb{K}^\xi(\mathbb{M}_\mu^{\beta_{1,k}}(\rho_k))), \tag{4.21}$$

and we clearly have $Q_1 \geq 0$.

Using $V_\epsilon(\mathbb{K}^0(\rho)) = V_\epsilon(\rho)$ for any $\rho \in \mathcal{D}$ and $\sum_\mu p_{\mu,\rho_k}^{\beta_{1,k}} V_\epsilon(\mathbb{K}^0(\mathbb{M}_\mu^{\beta_{1,k}}(\rho_k))) = V_\epsilon(\mathbb{K}^{\beta_{1,k}}(\rho_k)) - Q_2(\chi_k)$ with

$$Q_2(\chi) = \tfrac{1}{4}\sum_{n=1}^{d}\sum_{\mu,\nu} \mathrm{Tr}\left(M_\mu^{\beta_1}\rho M_\mu^{\beta_1\dagger}\right)\mathrm{Tr}\left(M_\nu^{\beta_1}\rho M_\nu^{\beta_1\dagger}\right) \left(\frac{\langle n|\, M_\mu^{\beta_1}\rho M_\mu^{\beta_1\dagger}\,|n\rangle}{\mathrm{Tr}\left(M_\mu^{\beta_1}\rho M_\mu^{\beta_1\dagger}\right)} - \frac{\langle n|\, M_\nu^{\beta_1}\rho M_\nu^{\beta_1\dagger}\,|n\rangle}{\mathrm{Tr}\left(M_\nu^{\beta_1}\rho M_\nu^{\beta_1\dagger}\right)}\right)^2, \tag{4.22}$$

we have

$$\mathbb{E}\left[W_\epsilon(\chi_{k+1})|\chi_k, u_k\right] = W_\epsilon(\chi_k) - Q_1(\chi_k) - Q_2(\chi_k), \tag{4.23}$$

where the continuous and non-negative functions Q_1 and Q_2 are defined by (4.21) and (4.22). $W_\epsilon(\chi_k)$ is thus a supermartingale. According to Theorem A.4, the ω-limit set I_∞ of χ_k satisfies

$$I_\infty \subset \left\{(\rho, \beta_1) \in \mathcal{D} \times [-\bar{u}, \bar{u}] \mid Q_1(\rho, \beta_1) = 0,\ Q_2(\rho, \beta_1) = 0\right\}.$$

We now show in three steps that I_∞ only consists of the target pure state $|\bar{n}\rangle\langle\bar{n}|$.

Step 1. For all $\delta > 0$, there exists $\bar{u} > 0$ small enough such that

$$I_\infty \subset \bigcup_{n=1}^{d}\Big\{(\rho, \beta_1) \in \mathcal{D} \times [-\bar{u}, \bar{u}] \mid \langle n|\rho|n\rangle \geq 1 - \delta\Big\}.$$

Proof of step 1. Step 1 follows from $Q_2(\rho) = 0$ for all $(\rho, \beta_1) \in I_\infty$ and the following lemma.

Lemma 4.5. *Consider the function Q_2 defined by (4.22) and $\widetilde{u} > 0$. Then there exists $C > 0$ such that for all $(\rho, \beta_1) \in \mathcal{D} \times [-\widetilde{u}, \widetilde{u}]$ satisfying $Q_2(\rho, \beta_1) = 0$, there exists $n \in \{1, \ldots, d\}$ such that $\rho_{n,n} = \langle n|\,\rho\,|n\rangle \geq 1 - C|\beta_1|$.*

Proof of Lemma 4.5 The condition $Q_2 = 0$ implies that, for all n, μ, ν

$$\mathrm{Tr}\left(M_\nu^{\beta_1}\rho M_\nu^{\beta_1\dagger}\right)\langle n|\, M_\mu^{\beta_1}\rho M_\mu^{\beta_1\dagger}\,|n\rangle = \mathrm{Tr}\left(M_\mu^{\beta_1}\rho M_\mu^{\beta_1\dagger}\right)\langle n|\, M_\nu^{\beta_1}\rho M_\nu^{\beta_1\dagger}\,|n\rangle.$$

Taking the sum over all ν, we get for all n and μ,

$$\langle n|\, M_\mu^{\beta_1}\rho M_\mu^{\beta_1\dagger}\,|n\rangle = \mathrm{Tr}\left(M_\mu^{\beta_1}\rho M_\mu^{\beta_1\dagger}\right)\langle n|\,\mathbb{K}^{\beta_1}(\rho)\,|n\rangle\,.$$

Since $M_\mu^{\beta_1}$ and $\mathbb{K}^{\beta_1}$ are C^2 function of β_1, these relations read ($\rho_{n',n'} = \langle n'|\,\rho\,|n'\rangle$)

$$|c_{\mu,n}|^2\rho_{n,n} = \left(\sum_{n'}|c_{\mu,n'}|^2\rho_{n',n'}\right)\rho_{n,n} + \beta_1\; b_{\mu,n}(\rho,\beta_1), \tag{4.24}$$

where the scalar functions $b_{\mu,n}$ depend continuously on ρ and β_1.

Let us prove the lemma by contradiction. Assume that for all $C > 0$, there exists $(\rho^C, \beta_1^C) \in \mathcal{D}\times[-\widetilde{u},\widetilde{u}]$ satisfying $Q_2(\rho^C,\beta_1^C) = 0$, such that

$$\forall n \in \{1,\ldots,d\},\;\; \rho_{n,n}^C \leq 1 - C|\beta_1^C|.$$

Take C tending towards $+\infty$. Since ρ^C and β_1^C remain in a compact set, we can assume, up to some extraction process, that ρ^C and β_1^C converge towards ρ^* and β_1^* in $\mathcal{D}$ and $[-\widetilde{u},\widetilde{u}]$. Since $|\beta_1^C| \leq (1-\rho_{n,n}^C)/C \leq 1/C$, we have $\beta_1^* = 0$. Since

$$|c_{\mu,n}|^2\rho_{n,n}^C = \left(\sum_{n'}|c_{\mu,n'}|^2\rho_{n',n'}^C\right)\rho_{n,n}^C + \beta_1^C\; b_{\mu,n}(\rho^C,\beta_1^C), \tag{4.25}$$

we have by continuity for $C \to +\infty$

$$|c_{\mu,n}|^2\rho_{n,n}^* = \left(\sum_{n'}|c_{\mu,n'}|^2\rho_{n',n'}^*\right)\rho_{n,n}^*,$$

for all n and μ. Thus, there exists $n^* \in \{1,\ldots,d\}$ such that $\rho^* = |n^*\rangle\langle n^*|$ (see the proof of Theorem 4.4). Since $\rho_{n^*,n^*}^* = 1$, for C large enough, $\rho_{n^*,n^*}^C > 1/2$ and thus

$$\left(\sum_{n'}|c_{\mu,n'}|^2\rho_{n',n'}^C\right) = |c_{\mu,n^*}|^2 - \beta_1^C\;\frac{b_{\mu,n^*}(\rho^C,\beta_1^C)}{\rho_{n^*,n^*}^C}. \tag{4.26}$$

Taking $n \neq n^*$, by Assumption 4.4, there exists μ such that $|c_{\mu,n}|^2 \neq |c_{\mu,n^*}|^2$. Replacing (4.26) in (4.25) yields

$$\left(|c_{\mu,n}|^2 - |c_{\mu,n^*}|^2 + \beta_1^C\;\tfrac{b_{\mu,n^*}(\rho^C,\beta_1^C)}{\rho_{n^*,n^*}^C}\right)\rho_{n,n}^C = \beta_1^C\; b_{\mu,n}(\rho^C,\beta_1^C).$$

Thus, there exists $C_0 > 0$ such that for $n \neq n^*$ and C large enough

$$\rho_{n,n}^C \leq C_0|\beta_1^C|.$$

But $\rho_{n^*,n^*}^C = 1 - \sum_{n\neq n^*}\rho_{n,n}^C \geq 1 - C_0(d-1)|\beta_1^C|$. This is in contradiction with $\rho_{n^*,n^*}^C \leq 1 - C|\beta_1^C|$ as soon as $C > C_0(d-1)$.

Step 2. There exist $\delta' > 0$ and $\bar{u} > 0$ small enough, such that

$$I_\infty \subset \{(\rho, \beta_1) \in \mathcal{D} \times [-\bar{u}, \bar{u}] \mid \langle \bar{n}|\rho|\bar{n}\rangle \geq 1 - \delta'\}.$$

Proof of step 2. If $Q_1(\rho, \beta_1) = 0$, the minimum of the function

$$[-\bar{u}, \bar{u}] \ni \xi \mapsto F_{\rho,\beta_1}(\xi) = \sum_\mu p^{\beta_1}_{\mu,\rho} V_\epsilon(\mathbb{K}^\xi(\mathbb{M}^{\beta_1}_\mu(\rho)))$$

is attained at $\xi = 0$.

By construction of V_ϵ, we know that for $\beta_1 = 0$, $\rho = |n\rangle\langle n|$,

$$\left.\frac{d^2 F_{|n\rangle\langle n|,0}(\xi)}{d\xi^2}\right|_{\xi=0} < 0,$$

if $n \neq \bar{n}$, and $\epsilon \leq \min_{n\neq\bar{n}}\left(\frac{\lambda_n}{R_{n,n}}\right)$. The dependence of $d^2F_{\rho,\beta_1}/d\xi^2$ versus ρ and β_1 is continuous from Assumption 4.5. Therefore, there exist $\delta', \bar{u} > 0$ such that for all

$$(\rho, \beta_1) \in \bigcup_{n\neq\bar{n}} \{(\rho, \beta_1) \in \mathcal{D} \times [-\bar{u}, \bar{u}] \mid \langle n|\rho|n\rangle \geq 1 - \delta'\} \tag{4.27}$$

we have

$$\left.\frac{d^2 F_{\rho,\beta_1}(\xi)}{d\xi^2}\right|_{\xi=0} < 0.$$

Therefore, for all (ρ, β_1) satisfying (4.27), $\xi = 0$ cannot minimize $F_{\rho,\beta_1}(\xi)$ and we have

$$I_\infty \cap \bigcup_{n\neq\bar{n}} \{(\rho, \beta_1) \in \mathcal{D} \times [-\bar{u}, \bar{u}] \mid \langle n|\rho|n\rangle \geq 1 - \delta'\} = \emptyset.$$

Therefore, we get

$$I_\infty \subset \{(\rho, \beta_1) \in \mathcal{D} \times [-\bar{u}, \bar{u}] \mid \langle \bar{n}|\rho|\bar{n}\rangle \geq 1 - \delta'\},$$

by setting $\delta = \delta'$ in Step 1, for $\bar{u}$ small enough. This finishes the proof of Step 2.

Step 3. $\bar{u}$ can be chosen small enough so that $I_\infty = \{(|\bar{n}\rangle\langle\bar{n}|, 0)\}$.

Proof of step 3. By construction of V_ϵ, we have for $\rho = |\bar{n}\rangle\langle\bar{n}|$ and $\beta_1 = 0$,

$$\left.\frac{d^2 F_{|\bar{n}\rangle\langle\bar{n}|,0}(\xi)}{d\xi^2}\right|_{\xi=0} > 0.$$

By continuity of $d^2F_{\rho,\beta_1}/d\xi^2$ with respect to ρ and β_1, we can choose $\bar{\delta}$ and $\bar{u}$ small enough such that for all $\beta_1 \in [-\bar{u}, \bar{u}]$ and ρ satisfying $\langle\bar{n}|\rho|\bar{n}\rangle \geq 1 - \bar{\delta}$, we have

$$\left.\frac{d^2 F_{\rho,\beta_1}(\xi)}{d\xi^2}\right|_{\xi=0} > 0.$$

This implies that on the domain

$$\{(\rho, \beta_1, \xi) \in \mathcal{D} \times [-\bar{u}, \bar{u}]^2 \mid \langle n|\rho|n\rangle \geq 1 - \bar{\delta}\},$$

F is a uniformly strongly convex function of $\xi \in [-\bar{u}, \bar{u}]$. Thus the argument of its minimum over $\xi \in [-\bar{u}, \bar{u}]$ is a continuous function of ρ and β_1.

Now we choose $\delta'' = \min\{\bar{\delta}/2, \delta'/2\}$, where δ' is as in step 2. Then, taking a convergent subsequence of $\{\chi_k\}_{k=1}^{\infty}$ (that we still denote by χ_k for simplicity sakes), we have

$$\chi_k \longrightarrow (\rho_\infty, \beta_{1,\infty}) \in I_\infty \subset \{(\rho, \beta_1) \in \mathcal{D} \times [-\bar{u}, \bar{u}] \mid \langle n|\rho|n\rangle \geq 1 - \bar{\delta}\}.$$

Therefore,

$$u_k = \underset{\xi \in [-\bar{u}, \bar{u}]}{\operatorname{argmin}} F_{\rho_k, \beta_{1,k}}(\xi) \to \underset{\xi \in [-\bar{u}, \bar{u}]}{\operatorname{argmin}} F_{\rho_\infty, \beta_{1,\infty}}(\xi) = 0.$$

In the above, we get the last equality from the fact that $Q_1(\rho_\infty, \beta_{1,\infty}) = 0$.

Since $\beta_{1,k} = u_{k-1}$, $\beta_{1,k}$ tends almost surely towards 0. We know that $Q_2(\rho_k, \beta_{1,k})$ and $Q_1(\rho_k, \beta_{1,k})$ tend also almost surely to 0 (Theorem A.4) and by uniform (in ρ) continuity with respect to $\beta_{1,k}$, $Q_2(\rho_k, 0)$ and $Q_1(\rho_k, 0)$ tend almost surely to 0. Since $Q_1(\rho, 0) = Q_2(\rho, 0) = 0$ implies $\rho = |\bar{n}\rangle\langle\bar{n}|$, ρ_k converges almost surely towards $|\bar{n}\rangle\langle\bar{n}|$. This completes the proof for $\tau = 1$.

Extension to $\tau > 1$. For $\tau > 1$ and $\chi = (\rho, \beta_1, \ldots, \beta_\tau)$, the proof is very similar. We still have (4.23) with Q_1 and Q_2 given by formulae analogous to (4.21) and (4.22): $\beta_{1,k}$ is replaced by $\beta_{\tau,k}$; $\mathbb{M}_\mu^{\beta_{1,k}}(\rho_k)$ is replaced by $\mathbb{K}^{\beta_{1,k}} \ldots \mathbb{K}^{\beta_{\tau-1,k}}(\mathbb{M}_\mu^{\beta_{\tau,k}}(\rho_k))$; $p_{\mu,\rho_k}^{\beta_{\tau,k}} = \operatorname{Tr}\left(M_\mu^{\beta_{\tau,k}} \rho_k M_\mu^{\beta_{\tau,k}\dagger}\right)$ is replaced by $\operatorname{Tr}\left(\mathbb{K}^{\beta_{1,k}} \ldots \mathbb{K}^{\beta_{\tau-1,k}}(M_\mu^{\beta_{\tau,k}} \rho_k M_\mu^{\beta_{\tau,k}\dagger})\right)$ (Kraus maps are trace preserving). The analogue of technical Lemma 4.5, whose proof relies on similar continuity and compactness arguments, has now the following statement:

Lemma 4.6. *There exists $C > 0$ such that for all $(\rho, \beta_1, \ldots, \beta_\tau) \in \mathcal{D} \times [-\tilde{u}, \tilde{u}]^\tau$ satisfying $Q_2(\rho, \beta_1, \ldots, \beta_\tau) = 0$, there exists $n \in \{1, \ldots, d\}$ such that $\rho_{n,n} \geq 1 - C(|\beta_1| + |\beta_2| + \ldots + |\beta_\tau|)$.*

The last part of the proof showing that, for $\bar{u}$ small enough, the control u_k is a continuous function of χ_k when χ_k is in the neighborhood of the ω-limit set

$$\{\chi \in \mathcal{D} \times [-\bar{u}, \bar{u}]^\tau \mid Q_1(\chi) = Q_2(\chi) = 0\},$$

remains almost the same. □

4.3.3 Quantum filter and separation principle

The natural estimation of the hidden state ρ satisfies the following dynamics

$$\rho_{k+1}^e = \mathbb{M}_{\mu_k}^{u_{k-\tau}}(\rho_k^e), \tag{4.28}$$

where the measurement outcome μ_k is driven by (4.18). In practice, the control u_k defined in Theorem 4.7 could only depend on this estimation ρ_k^e replacing ρ_k in χ_k. We have the following theorem which ensures the convergence of ρ_k and ρ_k^e towards the target state $|\bar{n}\rangle\langle\bar{n}|$ even though the initial states ρ_0 and ρ_0^e do not coincide.

Theorem 4.8. *Consider the recursive dynamics of state ρ_k obeying (4.18) and assume that the assumptions of Theorem 4.7 are satisfied. For each measurement outcome μ_k given by (4.18), consider the estimation ρ_k^e given by (4.28) with an initial condition ρ_0^e. Set $u_k = \mho(\rho_k^e, u_{k-1}, \dots, u_{k-\tau})$, where $\mho$ is given by (4.20). Then, there exists $u^* > 0$ such that for all $\bar{u} \in]0, u^*]$ and $\epsilon \in \left]0, \min_{n\neq\bar{n}}\left(\frac{\lambda_n}{R_{n,n}}\right)\right]$, ρ_k and ρ_k^e converge almost surely towards the target state $|\bar{n}\rangle\langle\bar{n}|$ as soon as $\ker(\rho_0^e) \subset \ker(\rho_0)$.*

Proof. We show that $\mathbb{E}\left[\operatorname{Tr}(\rho_k\bar{\rho})\,|\rho_0, \rho_0^e\right]$ (where we take the expectation over all jump realizations) depends linearly on ρ_0 even though we apply the feedback control.

Since $u_{k-\tau}$ is a function of $(\rho_0^e, \mu_0, \dots, \mu_{k-\tau-1})$, we have

$$\mathbb{E}\left[\operatorname{Tr}(\rho_k\bar{\rho})\,|\rho_0, \rho_0^e\right] = \sum_{\mu_0,\dots,\mu_{k-1}} \operatorname{Tr}\left(\widetilde{M}_{\mu_{k-1}}^{u_{k-\tau}}\left(\dots\left(\widetilde{M}_{\mu_0}^{u_{-\tau}}(\rho_0)\right)\right)\bar{\rho}\right),$$

where $\widetilde{M}_\mu^u(\rho) = M_\mu^u \rho M_\mu^{u\dagger}$. The linearity of $\mathbb{E}\left[\operatorname{Tr}(\rho_k\bar{\rho})\,|\rho_0, \rho_0^e\right]$ with respect to ρ_0 is thus clear. The rest of the proof will be done in the same way as the one presented for Theorem 3.3.

□

4.4 Imperfect measurements

We now consider the feedback control problem in the presence of classical measurement imperfections with the possibility of detection errors. We get the model described in Section 2.7. The imperfections in the measurement process are described by a classical probabilistic model relying on a left stochastic matrix $(\eta_{\mu',\mu})$, $\mu' \in \{1, \dots, m'\}$ and $\mu \in \{1, \dots, m\}$: $\eta_{\mu',\mu} \geq 0$ and for any μ, $\sum_{\mu'=1}^{m'} \eta_{\mu',\mu} = 1$. The integer m' corresponds to the number of imperfect outcomes and $\eta_{\mu',\mu}$ is the probability of having the imperfect outcome μ' knowing that the perfect one is μ.

Set

$$\widehat{\rho}_k = \mathbb{E}\left[\rho_k|\rho_0, \mu'_0, \dots, \mu'_{k-1}, u_{-\tau}, \dots, u_{k-\tau-1}\right]. \tag{4.29}$$

Since ρ_k follows (4.18), $\widehat{\rho}_k$ is also governed by a Markov process [90]:

$$\widehat{\rho}_{k+1} = \mathbb{L}_{\mu'_k}^{u_{k-\tau}}(\widehat{\rho}_k), \tag{4.30}$$

where

- for each μ', $\mathbb{L}^u_{\mu'}$ is the superoperator defined by

$$\widehat{\rho} \mapsto \mathbb{L}^u_{\mu'}(\widehat{\rho}) = \frac{\mathbf{L}^u_{\mu'}(\widehat{\rho})}{\mathrm{Tr}\left(\mathbf{L}^u_{\mu'}(\widehat{\rho})\right)}$$

  with $\mathbf{L}^u_{\mu'}(\widehat{\rho}) = \sum_{\mu=1}^m \eta_{\mu',\mu} M^u_\mu \widehat{\rho} M^{u\dagger}_\mu$;

- μ'_k is a random variable taking values μ' in $\{1, \dots, m'\}$ with probability

$$p^{u_{k-\tau}}_{\mu',\widehat{\rho}_k} = \mathrm{Tr}\left(\mathbf{L}^{u_{k-\tau}}_{\mu'}(\widehat{\rho}_k)\right).$$

Since $\eta_{\mu',\mu}$ is a left stochastic matrix, we have

$$\mathbb{E}\left[\widehat{\rho}_{k+1} | \widehat{\rho}_k = \rho, u_{k-\tau} = u\right] = \sum_{\mu'=1}^{m'} \mathbf{L}^u_{\mu'}(\widehat{\rho}) = \sum_{\mu=1}^{m} M^u_\mu \widehat{\rho} M^{u\dagger}_\mu = \mathbb{K}^u(\widehat{\rho}),$$

which is precisely the Kraus map (4.12) associated with the Markov process ρ_k. By Assumption 4.3, each pure state $|n\rangle \langle n|$, for $n \in \{1, \dots, d\}$ remains a fixed point of the Markov process (4.30), when $u \equiv 0$.

We now consider the dynamics of the filter state $\widehat{\rho}$ in the presence of delays in the feedback control. Similar to the case with perfect measurements, let $\widehat{\chi}_k = (\widehat{\rho}_k, \beta_{1,k}, \dots, \beta_{\tau,k})$ be the filter state at step k, where $\beta_{l,k} = u_{k-l}$ is the feedback control at time step k delayed l steps. Then the delay dynamics are determined by the following Markov chain

$$\begin{cases} \widehat{\rho}_{k+1} &= \mathbb{L}^{\beta_{\tau,k}}_{\mu'_k}(\widehat{\rho}_k) \\ \beta_{1,k+1} &= u_k \\ \beta_{2,k+1} &= \beta_{1,k} \\ &\vdots \\ \beta_{\tau,k+1} &= \beta_{\tau-1,k}. \end{cases} \tag{4.31}$$

Instead of Assumption 4.4 we now assume

Assumption 4.6. *For all $n_1 \neq n_2$ in $\{1, \dots, d\}$, there exists $\mu' \in \{1, \dots, m'\}$ such that*

$$\sum_{\mu=1}^m \eta_{\mu',\mu} |c_{\mu,n_1}|^2 \neq \sum_{\mu=1}^m \eta_{\mu',\mu} |c_{\mu,n_2}|^2.$$

Assumption 4.6 means that there exists a μ' such that the statistics when $u \equiv 0$ for obtaining the measurement result μ' are different for the fixed points $|n_1\rangle \langle n_1|$ and $|n_2\rangle \langle n_2|$. This follows by noting that $\mathrm{Tr}\left(\mathbf{L}^0_{\mu'}(|n\rangle \langle n|)\right) = \sum_{\mu=1}^m \eta_{\mu',\mu} |c_{\mu,n}|^2$ for $n \in \{1, \dots, d\}$.

We now state the main result of this section, which is the analogue of Theorem 4.7 in the case of imperfect measurements.

Theorem 4.9. *Consider the Markov chain (4.31) with Assumptions 4.3, 4.5 and 4.6. Take $\bar{n} \in \{1, \dots, d\}$. Assume that the directed graph G associated to the Metzler matrix R of Lemma 4.3 is strongly connected. Take $\epsilon > 0$, $\sigma \in \mathbb{R}_+^d$ solution of $R\sigma = \lambda$ with $\sigma_{\bar{n}} = 0$, , $\lambda_n < 0$ for $n \in \{1, \dots, d\} \setminus \{\bar{n}\}$, $\lambda_{\bar{n}} = -\sum_{n \neq \bar{n}} e_n \lambda_n$ (see Lemma 4.4) and set*

$$V_\epsilon(\widehat{\rho}) = \sum_{n=1}^{d} \sigma_n \langle n| \widehat{\rho} |n\rangle - \tfrac{\epsilon}{2}(\langle n| \widehat{\rho} |n\rangle)^2.$$

Take $\bar{u} > 0$ and consider the following feedback law

$$u_k = \underset{\xi \in [-\bar{u}, \bar{u}]}{argmin} \big(\mathbb{E}\left[W_\epsilon(\widehat{\chi}_{k+1}) | \widehat{\chi}_k, u_k = \xi\right]\big) = \mho(\widehat{\chi}_k) \tag{4.32}$$

where $W_\epsilon(\widehat{\chi}) = V_\epsilon(\mathbb{K}^{\beta_1}(\mathbb{K}^{\beta_2}(\dots\dots\mathbb{K}^{\beta_\tau}(\widehat{\rho})\dots)))$.

Then there exists $u^* > 0$ such that for all $\bar{u} \in]0, u^*]$ and $\epsilon \in \left]0, \min_{n \neq \bar{n}} \left(\frac{\lambda_n}{R_{n,n}}\right)\right]$, the closed-loop Markov chain of state $\widehat{\chi}_k$ with the feedback law (4.32) converges almost surely towards $(|\bar{n}\rangle \langle \bar{n}|, 0, \dots, 0)$, for any initial condition $\widehat{\chi}_0 \in \mathcal{D} \times [-\bar{u}, \bar{u}]^\tau$.

The proof of this theorem is almost identical to that of Theorem 4.7 with $\mathbb{M}_\mu^u$ replaced by $\mathbb{L}_{\mu'}^u$ and $p_{\mu,\rho}^u$ replaced by $p_{\mu',\widehat{\rho}}^u$ and we do not give the details of this proof.

For estimating the hidden state $\widehat{\rho}$ necessary to apply the feedback (4.32), let us consider the estimate $\widehat{\rho}^e$ given by

$$\widehat{\rho}_{k+1}^e = \mathbb{L}_{\mu'_k}^{u_{k-\tau}}(\widehat{\rho}_k^e), \tag{4.33}$$

where μ'_k corresponds to the imperfect outcome detected at step k. Such μ'_k is correlated to the perfect and hidden outcome μ_k of (4.18) through the classical stochastic process attached to $(\eta_{\mu',\mu})$: for each μ_k, μ'_k is a random variable which is equal to $\mu' \in \{1, \dots, m'\}$, with probability η_{μ',μ_k}.

In practice, the control u_k defined in Theorem 4.9 could only depend on the estimation $\widehat{\rho}_k^e$ replacing $\widehat{\rho}_k$ in $\widehat{\chi}_k = (\widehat{\rho}_k, u_{k-1}, \dots, u_{k-\tau})$. The following result guaranties the convergence of the feedback scheme when $\ker(\widehat{\rho}_0^e) \subset \ker(\rho_0)$.

Theorem 4.10. *Consider the recursive dynamics of state ρ_k obeying (4.18) and take the assumptions of Theorem 4.9. Consider the estimation $\widehat{\rho}_k^e$ given by (4.33) with an initial condition $\widehat{\rho}_0^e$. Set $u_k = \mho(\widehat{\rho}_k^e, u_{k-1}, \dots, u_{k-\tau})$ where $\mho$ is given by (4.32).*
Then there exists $u^* > 0$ such that for all $\bar{u} \in]0, u^*]$ and $\epsilon \in \left]0, \min_{n \neq \bar{n}} \left(\frac{\lambda_n}{R_{n,n}}\right)\right]$, ρ_k and $\widehat{\rho}_k^e$ converge almost surely towards the target state $|\bar{n}\rangle \langle \bar{n}|$ as soon as $\ker(\widehat{\rho}_0^e) \subset \ker(\rho_0)$.

Proof. Let us first prove that $\widehat{\rho}_k$ defined by (4.29) converges almost surely towards $|\bar{n}\rangle \langle \bar{n}|$. Since $\widehat{\rho}_0 = \rho_0$, we have $\ker(\widehat{\rho}_0^e) \subset \ker(\widehat{\rho}_0)$. Thus, there exist $\widehat{\rho}_0^c \in \mathcal{D}$ and $\gamma \in]0, 1[$, such that $\widehat{\rho}_0^e = \gamma \widehat{\rho}_0 + (1 - \gamma)\widehat{\rho}_0^c$. Similarly to the proof of Theorem 4.8, $\mathbb{E}\left[\mathrm{Tr}\left(\widehat{\rho}_k \bar{\rho}\right) | \widehat{\rho}_0, \widehat{\rho}_0^e\right]$ depends linearly on $\widehat{\rho}_0$:

$$\mathbb{E}\left[\mathrm{Tr}\left(\widehat{\rho}_k \bar{\rho}\right) \mid \widehat{\rho}_0^e, \widehat{\rho}_0^e\right] = \gamma \mathbb{E}\left[\mathrm{Tr}\left(\widehat{\rho}_k \bar{\rho}\right) \mid \widehat{\rho}_0, \widehat{\rho}_0^e\right] + (1 - \gamma)\mathbb{E}\left[\mathrm{Tr}\left(\widehat{\rho}_k \bar{\rho}\right) \mid \widehat{\rho}_0^c, \widehat{\rho}_0^e\right].$$

Moreover, by Theorem 4.9, $\mathbb{E}\left[\mathrm{Tr}\left(\widehat{\rho}_k\bar{\rho}\right) \mid \widehat{\rho}^e_0, \widehat{\rho}^e_0\right]$ converges almost surely towards one, thus, $\mathbb{E}\left[\mathrm{Tr}\left(\widehat{\rho}_k\bar{\rho}\right) \mid \widehat{\rho}_0, \widehat{\rho}^e_0\right]$ and $\mathbb{E}\left[\mathrm{Tr}\left(\widehat{\rho}_k\bar{\rho}\right) \mid \widehat{\rho}^c_0, \widehat{\rho}^e_0\right]$ converge also towards one. Consequently, $\widehat{\rho}_k$ converges almost surely towards $|\bar{n}\rangle\langle\bar{n}|$.

Since $\widehat{\rho}_k$ is the conditional expectation of ρ_k knowing the past imperfect outcomes and control inputs, and since its limit $|\bar{n}\rangle\langle\bar{n}|$ is a pure state, ρ_k converges necessarily towards the same pure state almost surely. Convergence of $\widehat{\rho}^e_k$ relies on similar arguments exploiting the linearity of $\mathbb{E}\left[\mathrm{Tr}\left(\widehat{\rho}^e_k\bar{\rho}\right) | \widehat{\rho}_0, \widehat{\rho}^e_0\right]$ versus $\widehat{\rho}_0$.

□

4.5 The photon box

In this section, we reconsider the photon box presented in Chapter 3. Not only we suppose that there exists some delays in the measurement process, but we assume also that the measurements are imperfect. Indeed, we give the explicit expression of the feedback which is obtained by the Lyapunov design discussed in this chapter. Moreover, this feedback has been experimentally tested in Laboratoire Kastler Brossel (LKB) at Ecole Normal supérieure (ENS) de Paris and the result has been published in [88].

4.5.1 Experimental system

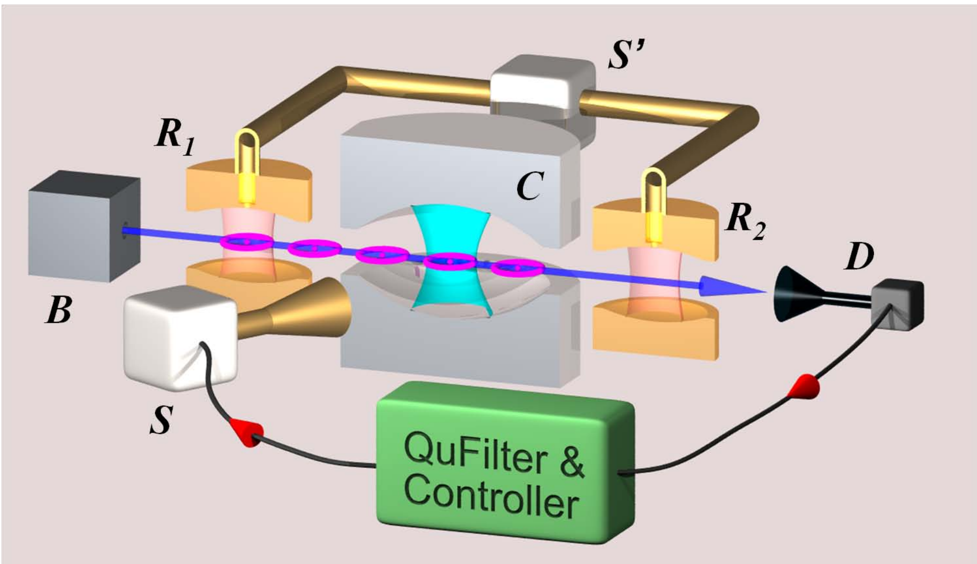

Figure 4.1: Scheme of the experimental setup.

The system to be controlled is a microwave electromagnetic field trapped in a cavity C (see Fig. 4.1), made of two high-reflection mirrors facing each other and able to store photons for almost $T_{\mathrm{cav}} \sim 0.1\,\mathrm{s}$. When prepared in a classical state, namely a coherent state $|\alpha\rangle$, where $\alpha \in \mathbb{C}$ denotes the amplitude of the field, the number of photons in the field is not well-defined: the photon number probability distribution $P(n_{\mathrm{ph}})$ is a Poisson law of parameter $\langle n_{\mathrm{ph}}\rangle = |\alpha|^2$. Coherent states are robust states that remain coherent while their energy $\langle n_{\mathrm{ph}}\rangle$ decays with a characteristic time T_{cav}. The non-classical photon-number states $|n = n_{\mathrm{ph}} + 1\rangle \equiv |n_{\mathrm{ph}}\rangle$, or Fock states containing exactly $n_{\mathrm{ph}} \in \mathbb{N}$ photons, are more fragile: a so-called quantum jump towards $|n_{\mathrm{ph}} - 1\rangle$ occurs on average after time $T_{\mathrm{cav}}/n_{\mathrm{ph}}$ only. Real-time information is then required to correct for these jumps and stabilize these states.

In LKB experiment, this information is provided by individual two-level atoms, whose states are denoted $|g\rangle$ and $|e\rangle$, and which are prepared in box B and sent through the cavity mode every time interval T_p. In additional cavities R_1 and R_2, we can map these states to and from their quantum superposition which is sensitive to the field in C. While interacting with the cavity field, the atoms act as atomic clocks whose rate would be proportional to the number n_{ph} of photons stored in the cavity: they accumulate a phase $\phi(n_{\mathrm{ph}}) = \phi_0(n_{\mathrm{ph}} + 1/2)$, where ϕ_0 is the dephasing per photon. Measuring each atom in the $\{|g\rangle, |e\rangle\}$ basis by means of detector D gives partial information about n_{ph}. Since the atomic preparation is not deterministic (Poisson process with a mean number of atoms per sample of $\langle n_a\rangle \approx 0.6$), zero, one or two atoms may be detected.

A real-time computer then runs the quantum filter, which estimates the actual density matrix ρ of the field using the outcome of the measurement and all available knowledge on the experimental setup. Notably, it takes into account the limited detection efficiency ε (percentage of atoms that are actually detected) and the detection errors η_e and η_g ($\eta_{\mu\in\{e,g\}}$ is the probability to detect an atom in the wrong state).

The controller eventually calculates through its feedback law the amplitude $u \in \mathbb{R}$ (scalar control input) of the classical microwave field that is injected into the cavity by the classical source S so as to bring the cavity field closer to the target state. The succession of the atomic detection, the state estimation and the microwave injection define one iteration of our feedback loop. For a more detailed description see [88, 87, 44].

4.5.2 The controlled Markov process and quantum filter

The three main processes that drive the evolution of our system are decoherence, injection and measurement. Their action onto the system's state can be described by three superoperators $\mathbb{T}$, $\mathbb{D}$ and $\mathbb{P}$, respectively (see [36] for more details). All operators are expressed in the truncated Fock-basis $(|n_{\mathrm{ph}}\rangle)_{n_{\mathrm{ph}}=0,\ldots,n^{\max}}$ to $n^{\max}$ photons. With notations given in previous sections, $d = n^{\max} + 1$ and the index $n = n_{\mathrm{ph}} + 1$.

Decoherence

The decoherence manifests through spontaneous loss or capture of a photon to or from the environment. Thus, it can be sufficiently described by the action of the superoperator $\mathbb{T}$ in the justified approximation $\theta = T_{\mathrm{p}}/T_{\mathrm{cav}} \ll 1$ in the form of

$$\rho \mapsto \mathbb{T}_\theta(\rho) = L_0\rho L_0^\dagger + L_-\rho L_-^\dagger + L_+\rho L_+^\dagger. \tag{4.34}$$

Here, the three operators L_0, L_- and L_+ refer to the situations where the photon number changes by 0, -1 and $+1$ after the time interval T_{p}. They thus read

$$L_0 = \mathbb{1} - \theta(1/2 + n_{\mathrm{th}})\,\mathbf{N} - (\theta n_{\mathrm{th}}/2)\,\mathbb{1}, \tag{4.35}$$

$$L_- = \sqrt{\theta(1 + n_{\mathrm{th}})}\,\mathbf{a}, \tag{4.36}$$

$$L_+ = \sqrt{\theta n_{\mathrm{th}}}\,\mathbf{a}^\dagger, \tag{4.37}$$

where $\mathbf{a}$ and $\mathbf{a}^\dagger$ are photon annihilation and creation operators, ($\mathbf{a}\,|n_{\mathrm{ph}}\rangle = \sqrt{n_{\mathrm{ph}}}\,|n_{\mathrm{ph}}-1\rangle$ and $\mathbf{a}^\dagger\,|n_{\mathrm{ph}}\rangle = \sqrt{n_{\mathrm{ph}}+1}\,|n_{\mathrm{ph}}+1\rangle$), and $\mathbf{N} = \mathbf{a}^\dagger\mathbf{a}$ is the photon number operator ($\mathbf{N}\,|n_{\mathrm{ph}}\rangle = n_{\mathrm{ph}}\,|n_{\mathrm{ph}}\rangle$). Besides, n_{th} is the mean number of photons in the cavity mode at thermal equilibrium with its environment.

Injection

The control field is realized by injecting into the cavity a small coherent microwave field of real amplitude $u \in \mathbb{R}$. It is described by the unitary displacement operator $D_u = \exp(u\mathbf{a}^\dagger - u\mathbf{a})$. The evolution of the state ρ after the control injection is thus modeled through

$$\rho \mapsto \mathbb{D}_u(\rho) = D_u \rho D_{-u}. \tag{4.38}$$

Since all analysis is performed up to the second order of the control parameter u, we use the following approximation of the displaced state:

$$\mathbb{D}_u(\rho) \approx \rho - u[\rho, \mathbf{a}^\dagger - \mathbf{a}] + \frac{u^2}{2}[[\rho, \mathbf{a}^\dagger - \mathbf{a}], \mathbf{a}^\dagger - \mathbf{a}]. \tag{4.39}$$

In the following, we therefore limit the control to $u \in [-\bar{u}, \bar{u}]$ with $\bar{u} = 0.1 \ll 1$. Besides, this approximation allows to significantly reduce the computation time during the real-time state filtering.

Measurement

In the ideal case, every detection of an atomic sample gives one of the two possible results $\mu \in \{e, g\}$ and the system gets projected onto one of the two states given by

$$\mathbb{P}_\mu(\rho) = \frac{M_\mu \rho M_\mu^\dagger}{\mathrm{Tr}\left(M_\mu \rho M_\mu^\dagger\right)}. \tag{4.40}$$

The operators M_μ depend on the experimental settings, the relative phase ϕ_r between cavities R_1 and R_2 and the dephasing per photon ϕ_0:

$$M_g = \cos\left(\frac{\phi_r \mathbb{1} + \phi_0(\mathbf{N} + 1/2\mathbb{1})}{2}\right), \tag{4.41a}$$

$$M_e = \sin\left(\frac{\phi_r \mathbb{1} + \phi_0(\mathbf{N} + 1/2\mathbb{1})}{2}\right), \tag{4.41b}$$

$\mu' \setminus \mu$	$\varnothing$	g	e	gg	ee	ge or eg
$\varnothing$	1	$1-\varepsilon$	$1-\varepsilon$	$(1-\varepsilon)^2$	$(1-\varepsilon)^2$	$(1-\varepsilon)^2$
g	0	$\varepsilon(1-\eta_g)$	$\varepsilon\eta_e$	$2\varepsilon(1-\varepsilon)(1-\eta_g)$	$2\varepsilon(1-\varepsilon)\eta_e$	$\varepsilon(1-\varepsilon)(1-\eta_g+\eta_e)$
e	0	$\varepsilon\eta_g$	$\varepsilon(1-\eta_e)$	$2\epsilon(1-\varepsilon)\eta_g$	$2\varepsilon(1-\varepsilon)(1-\eta_e)$	$\varepsilon(1-\varepsilon)(1-\eta_e+\eta_g)$
gg	0	0	0	$\varepsilon^2(1-\eta_g)^2$	$\varepsilon^2\eta_e^2$	$\varepsilon^2\eta_e(1-\eta_g)$
ge	0	0	0	$2\varepsilon^2\eta_g(1-\eta_g)$	$2\varepsilon^2\eta_e(1-\eta_e)$	$\varepsilon^2((1-\eta_g)(1-\eta_e)+\eta_g\eta_e)$
ee	0	0	0	$\varepsilon^2\eta_g^2$	$\varepsilon^2(1-\eta_e)^2$	$\varepsilon^2\eta_g(1-\eta_e)$

Table 4.1: Stochastic matrix $\eta_{\mu',\mu}$ showing the probability to measure outcome μ' for each ideal measurement outcome μ. Note that we cannot technically distinguish detection outcomes $\mu' = \{ge\}$ and $\mu' = \{eg\}$.

(with notations given in Section 3.1, $\frac{1}{2}\phi_r + \frac{1}{4}\phi_0 = \varphi_0$ and $\frac{1}{2}\phi_0 = \vartheta$). These measurement operators are diagonal in the photon number basis $\{|n_{\text{ph}}\rangle\}$, illustrating their quantum non-demolition nature with respect to the photon number operator and thus fulfilling Assumption 4.3. Besides, Assumption 4.4 can also be fulfilled by a proper choice of the experimental parameters ϕ_r and ϕ_0

In the real experiment, the atom source is probabilistic and is characterized by a truncated Poisson probability distribution $P_a(n_a) \geq 0$ to have $n_a \in \{0, 1, 2\}$ atom(s) in a sample (we neglect events with more than two atoms). This expands the set of the possible detection outcomes to $m = 7$ values $\mu \in \{\varnothing, g, e, gg, eg, ge, ee\}$, related to the following measurement operators:

$$
\begin{aligned}
L_\varnothing &= \sqrt{P_a(0)}\,\mathbb{1}, \\
L_g &= \sqrt{P_a(1)}\,M_g, \\
L_e &= \sqrt{P_a(1)}\,M_e, \\
L_{gg} &= \sqrt{P_a(2)}\,M_g^2, \\
L_{ge} &= \sqrt{P_a(2)}\,M_g M_e, \\
L_{eg} &= \sqrt{P_a(2)}\,M_g M_e, \\
L_{ee} &= \sqrt{P_a(2)}\,M_e^2.
\end{aligned} \tag{4.42}
$$

Finally, the real measurement process is not perfect: the detection efficiency is limited to $\varepsilon < 1$ and the state detection errors are non-zero ($0 < \eta_{e/g} < 1$). These imperfections are taken into account by considering the left-stochastic matrix $\eta_{\mu',\mu}$ given in Table 4.1 [90]. Consequently, the optimal state estimate after measurement outcome μ' gets the following form

$$
\mathbb{P}_{\mu'}(\rho) = \frac{\sum_{\mu=1}^{m} \eta_{\mu',\mu} L_\mu \rho L_\mu^\dagger}{\mathrm{Tr}\left(\sum_{\mu=1}^{m} \eta_{\mu',\mu} L_\mu \rho L_\mu^\dagger\right)}. \tag{4.43}
$$

State estimation. The last experimental complication to be taken into account is the spatial separation of the cavity C and the detector D, which results in τ atomic samples flying between them at every moment. This introduces a delay in the control field since a sample measured at step k has been in the cavity just after control injection made at step $k-\tau$. After taking into account our full knowledge about the experiment, we finally get the following state estimate at step $k+1$:

$$\widehat{\rho}^e_{k+1} = \mathbb{P}_{\mu'_k}(\mathbb{D}_{u_{k-\tau}}(\mathbb{T}_\theta(\widehat{\rho}^e_k))) \equiv \mathbb{L}^{u_{k-\tau},\theta}_{\mu'_k}(\widehat{\rho}^e_k). \tag{4.44}$$

4.5.3 Feedback controller

For $\theta = 0$ (no cavity decoherence), the Markov model of density matrix $\widehat{\rho}$ associated to the filter (4.44) is exactly of the form (4.31) with $(|n_{\text{ph}}\rangle \langle n_{\text{ph}}|)_{n_{\text{ph}}=0,\dots,n^{\max}}$ being fixed-points in open-loop. Similarly, the underlying Markov process of state ρ, the true cavity state unobservable in practice because of detection imperfections and delays, admits the same fixed-points. With parameters given in Subsection 4.5.3 (except $\theta = 0$), these Markov processes satisfy Assumptions 4.3-4.4-4.5 for ρ and Assumptions 4.3-4.5-4.6 for $\widehat{\rho}$. Moreover the Metzler matrix R of Lemma 4.4 is irreducible since its directed graph coincides with the directed graph of $\mathbf{a}^\dagger - \mathbf{a}$, a tri-diagonal operator. Consequently the assumptions of Theorem 4.9 are satisfied. The feedback law proposed in Theorem 4.10 and relying on the filter state $\widehat{\rho}^e$ will stabilize globally the unobservable state ρ towards the goal photon-number state $|\bar{n}_{\text{ph}}\rangle \langle \bar{n}_{\text{ph}}|$. Numerous closed-loop simulations show that taking $\epsilon = 0$ in the feedback law does not destroy stability and does not affect the convergence rates. This explains why in the simulations and experiments, we set $\epsilon = 0$ despite the fact that Theorem 4.10 guaranties convergence only for arbitrary small but strictly positive ϵ.

For θ positive and small, the $|n_{\text{ph}}\rangle \langle n_{\text{ph}}|$'s are no more fixed-points for ρ and $\widehat{\rho}$ in open-loop. Nevertheless, closed-loop simulations and experimental data of figures 4.3 and 4.4 indicate that a slight adaptation of the feedback scheme of Theorem 4.10 steers and maintains ρ and $\widehat{\rho}$ close to the target $|\bar{n}_{\text{ph}}\rangle \langle \bar{n}_{\text{ph}}|$ between photon-number jumps induced by cavity decoherence (jump operators L_+ and L_- given in (4.35)). The feedback of Theorem 4.10 appears to be robust enough to compensate such cavity decoherence jumps induced by a finite lifetime of the photons.

Let us detail how to adapt the feedback scheme of Theorem 4.10. At each step of an ideal experiment, the control u_k minimizes the Lyapunov function $V_0(\tilde{\rho}_k) = \sum_{n_{\text{ph}}} \sigma_{n_{\text{ph}}} \langle n_{\text{ph}}| \tilde{\rho}_k |n_{\text{ph}}\rangle$ (ϵ is set to zero) calculated for state $\tilde{\rho}_k = \mathbb{D}_{u_k}(\rho_k)$. In our real experiment however, we also take into account decoherence and τ flying not-yet-detected samples and therefore choose u_k to minimize $V_0(\tilde{\rho}_k)$ for

$$\tilde{\rho}_k = \mathbb{D}_{u_k}(\mathbb{T}_\theta(\mathbb{K}^{u_{k-1},\theta}(\mathbb{K}^{u_{k-2},\theta}(\dots\mathbb{K}^{u_{k-\tau},\theta}(\widehat{\rho}^e_k)\dots)))),$$

with $\widehat{\rho}^e$ given by (4.44). Here we have introduced the Kraus map of the real experiment

$$\mathcal{D} \ni \rho \mapsto \mathbb{K}^{u,\theta}(\rho) = \sum_{\mu'=1}^{m'} \sum_{\mu=1}^{m} \eta_{\mu',\mu} L_\mu(\mathbb{D}_u(\mathbb{T}_\theta(\rho))) {L_\mu}^\dagger \in \mathcal{D}. \tag{4.45}$$

To simplify the minimization of W_0, we use approximation (4.39) and get

$$V_0(\mathbb{D}_u(\rho)) \approx V_0(\rho) - \left(a_1 u + a_2 \frac{u^2}{2}\right), \tag{4.46}$$

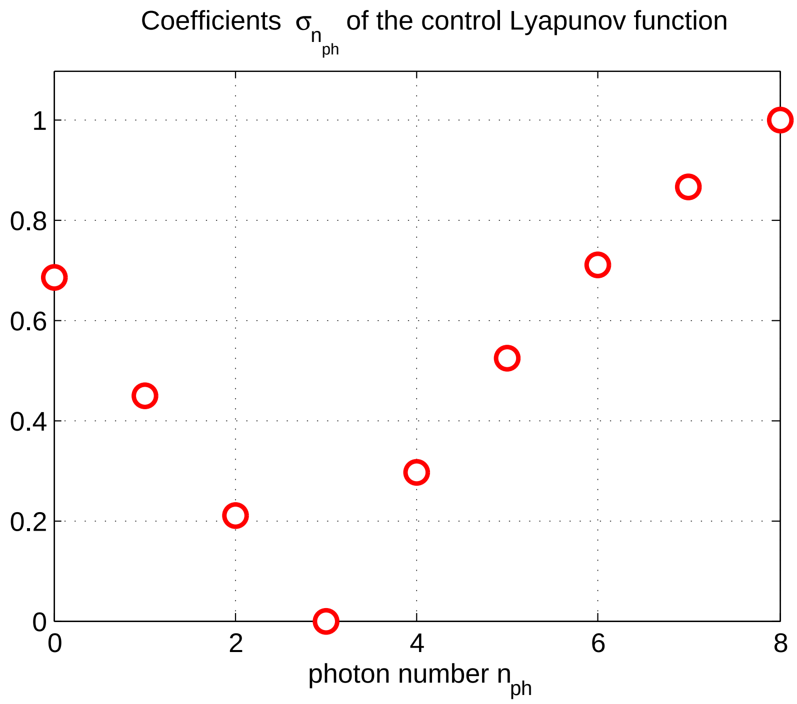

Figure 4.2: Coefficients of the Lyapunov function $V_0(\rho)$ used in the feedback for the simulations of Figure 4.3 and for the experiment of Figure 4.4.

with $a_1 = \operatorname{Tr}\left([\mathbf{a}^\dagger - \mathbf{a}, \sigma_{\mathbf{N}}]\rho\right)$ and $a_2 = \operatorname{Tr}\left([[\mathbf{a}^\dagger - \mathbf{a}, \sigma_{\mathbf{N}}], \mathbf{a}^\dagger - \mathbf{a}]\rho\right)$, where $\sigma_{\mathbf{N}}$ is the diagonal operator $\sum_{n_{\text{ph}}} \sigma_{n_{\text{ph}}} |n_{\text{ph}}\rangle \langle n_{\text{ph}}|$. The coefficients $\sigma_{n_{\text{ph}}}$ are computed using Lemma 4.4, where, for $n_{\text{ph}} \neq \bar{n}_{\text{ph}}$, $\lambda_{n_{\text{ph}}}$ are chosen strictly negative and with a decreasing modulus versus n_{ph} in order to compensate cavity decay. For $\bar{n}_{\text{ph}} = 3$, we have compared different setting in simulations and selected the profile displayed in Fig. 4.2. The control u minimizing V_0 is then approximated by

$$\underset{\xi \in [-\bar{u}, \bar{u}]}{\operatorname{argmax}} \left(a_1 \xi + a_2 \xi^2/2\right).$$

This approximated function takes its maximum necessary at one of the two extremes $-\bar{u}$, $\bar{u}$ or at the point $-a_1/a_2$ where its derivative is zero.

Simulations and experimental results. Closed-loop simulation of Figure 4.3 shows one typical Monte-Carlo trajectory of the feedback loop aiming on the stabilization of the 3-photon state $|\bar{n}_{\text{ph}} = 3\rangle$. The typical values of experimental parameters used in the simulations are the following: $n^{\max} = 8$, $\phi_0 = 0.245\,\pi$, $\phi_r = \pi/2 - \phi_0(\bar{n}_{\text{ph}} + 1/2)$, $\langle n_{\text{a}} \rangle = 0.6$, $\varepsilon = 0.35$, $\eta_e = 0.13$, $\eta_g = 0.11$, $T_{\text{p}} = 82\,\mu$s, $T_{\text{cav}} = 65$ ms, $n_{\text{th}} = 0.05$, and $\tau = 4$. For the feedback, $\sigma_{n_{\text{ph}}}$ are given in Figure 4.2, $\epsilon = 0$ and $\bar{u} = 1/10$. The initial states ρ_0 and $\widehat{\rho}_0^e$ coincide with the coherent state with $\bar{n}_{\text{ph}} = 3$ photons: $\mathbb{D}_{\sqrt{\bar{n}_{\text{ph}}}}(|0\rangle \langle 0|)$.

The results of the experimental implementation of the feedback scheme are presented in Figure 4.4. Figure 4.4(e) shows that the average fidelity of the target state is about

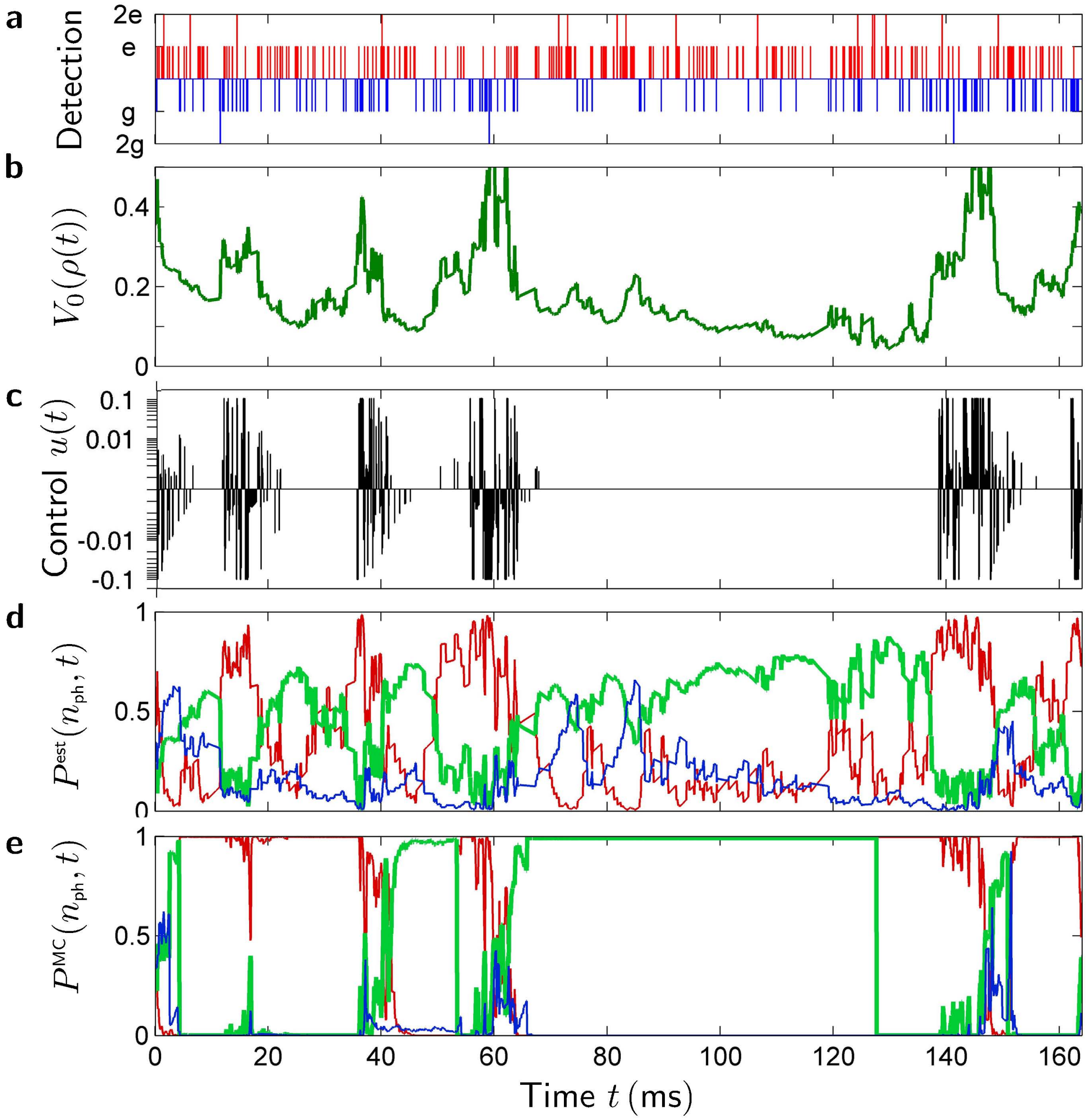

Figure 4.3: Simulation of one Monte-Carlo trajectory in closed-loop with the target state $|\bar{n}_{\text{ph}} = 3\rangle$. (a) Detection results. (b) control-Lyapunov function. (c) Control input. (d) Estimated photon number probabilities: $\sum_{n_{\text{ph}}<\bar{n}_{\text{ph}}} \langle n_{\text{ph}}| \widehat{\rho}^e |n_{\text{ph}}\rangle$ in red, $\langle \bar{n}_{\text{ph}}| \widehat{\rho}^e |\bar{n}_{\text{ph}}\rangle$ in thick green, and $\sum_{n_{\text{ph}}>\bar{n}_{\text{ph}}} \langle n_{\text{ph}}| \widehat{\rho}^e |n_{\text{ph}}\rangle$ in blue. (e) cavity photon number probabilities: $\sum_{n_{\text{ph}}<\bar{n}_{\text{ph}}} \langle n_{\text{ph}}| \rho |n_{\text{ph}}\rangle$ in red , $\langle \bar{n}_{\text{ph}}| \rho |\bar{n}_{\text{ph}}\rangle$ in thick green, and $\sum_{n_{\text{ph}}>\bar{n}_{\text{ph}}} \langle n_{\text{ph}}| \rho |n_{\text{ph}}\rangle$ in blue.

47%. Besides, the asymmetry between the distributions for $n_{\text{ph}} < \bar{n}_{\text{ph}}$ and $n_{\text{ph}} > \bar{n}_{\text{ph}}$ indicates the presence of quantum jumps occurring preferentially downwards ($\bar{n}_{\text{ph}} \to \bar{n}_{\text{ph}} - 1$). Contrarily to the simulations of Figure 4.3(e), the cavity photon numbers relied on ρ are not accessible in the experimental data of Figure 4.4, since we do not have access to the detection errors and to the cavity decoherence jumps. Nevertheless, green curves in simulation Figures 4.3(d) and 4.3(e) indicate that when $\langle \bar{n}_{\text{ph}} | \widehat{\rho}^e | \bar{n}_{\text{ph}} \rangle$ exceeds 8/10, ρ coincides, with high probability, with the goal state $|\bar{n}_{\text{ph}}\rangle \langle \bar{n}_{\text{ph}}|$.

Remark 5. *We remark that the feedback designed in this chapter is more efficient than the one proposed in [36] (the increase of* 10 *percent in fidelity).*

4.6 Conclusion

We have proposed the Lyapunov designs for state feedback stabilization of some discrete-time finite-dimensional quantum systems with QND measurements. Extensions of these designs are possible in different directions such as:

- replacing the continuous and one-dimensional input u by a multi-dimensional one $(u_1, \ldots, u_p)$;
- assuming that u belongs to a finite set of discrete values;
- taking an infinite dimensional state space as in [91], where the truncation to finite photon numbers is removed;
- considering continuous-time systems similar to the ones investigated in [71];
- ensuring convergence towards a sub-space instead of a pure-state and thus achieving a goal similar to error correction code as already proposed in [1].

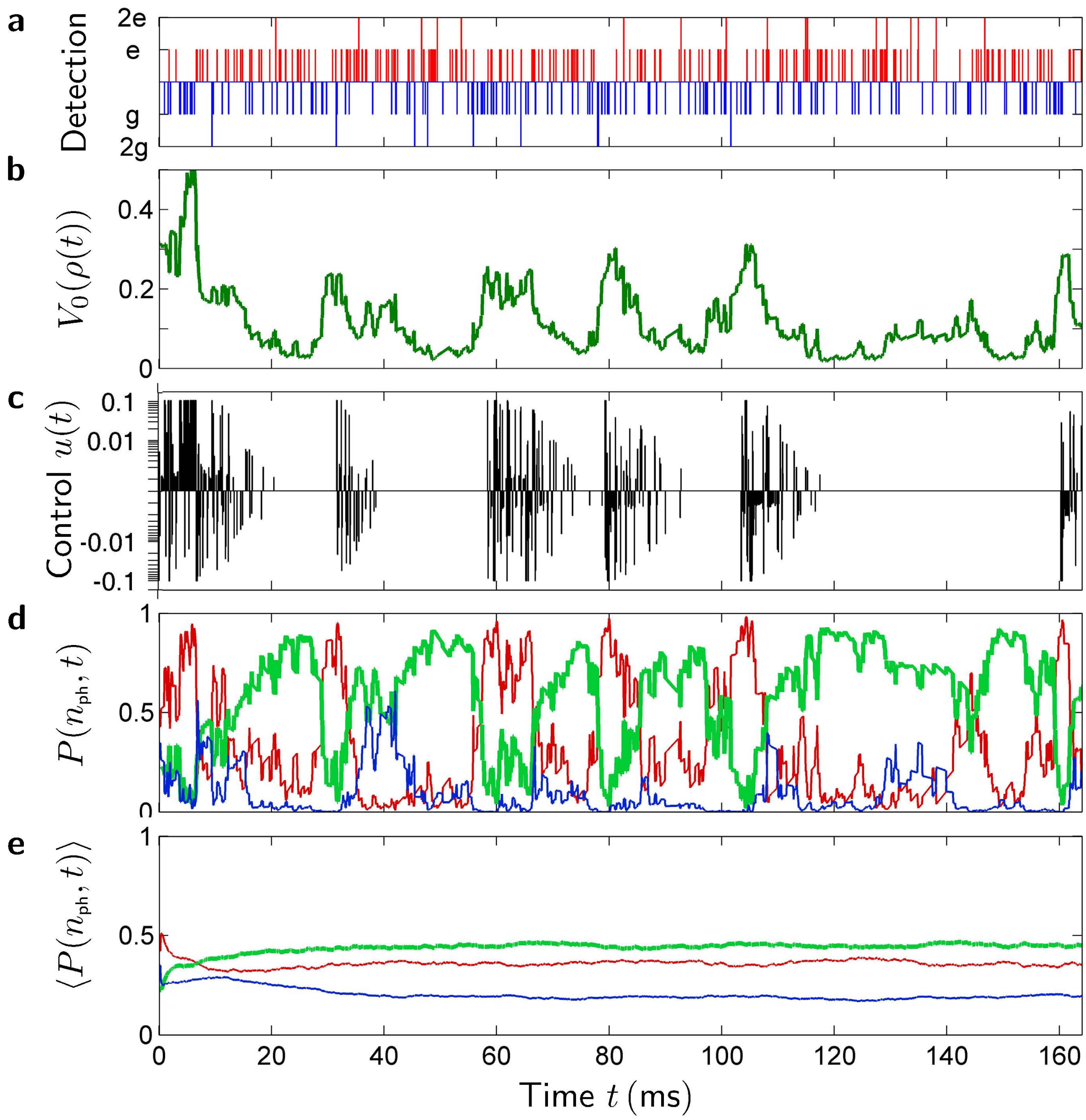

Figure 4.4: Single experimental trajectory of the feedback loop with the target state $|\bar{n}_{ph} = 3\rangle$. (a) Detection results. (b) Evolution of the control function. (c) Control injection. (d) Estimated photon number probabilities: $\sum_{n_{ph}<\bar{n}_{ph}} \langle n_{ph}| \widehat{\rho}^e |n_{ph}\rangle$ in red , $\langle \bar{n}_{ph}| \widehat{\rho}^e |\bar{n}_{ph}\rangle$ in thick green, and $\sum_{n_{ph}>\bar{n}_{ph}} \langle n_{ph}| \widehat{\rho}^e |n_{ph}\rangle$ in blue. (e) Estimated photon number probabilities averaged over 4000 experimental trajectories of the feedback loop.

Part II

Continuous-time open quantum system

Chapter 5

On stability of continuous-time quantum filters

Contents

Dans cette partie, nous considérons les systèmes quantiques ouverts en temps continu qui sont simultanément en interaction avec un environnement et subissent une certaine mesure. Ces systèmes sont généralement décrits par des équations différentielles stochastiques. Cette description peut être faite de deux manières : soit une équation de Schrödinger stochastique (SSE) [11, 26, 8] lorsque l'état quantique reste pur et que les sauts dus à l'environnement et la mesure sont pleinement intégrées, soit une équation maîtresse stochastique (SME) [9, 11] pour les états mixtes arbitraires lorsque seuls les sauts dus à la mesure sont pris en compte. En outre, ces équations différentielles stochastiques peuvent être caractérisées par deux types de bruits : les processus de Poisson apparaissent comme les bruits pour le cas sauts et les processus de Wiener pour les cas diffusifs. Les SMEs plus générales ou les SSEs sont simultanément dérivées par les deux termes de diffusion et de saut [10].

Dans ce chapitre, nous considérons le cas le plus fréquent des SMEs dérivées par un processus de Wiener et nous montrons que les filtres quantiques associés sont stables. Dans le prochain chapitre, nous allons présenter des équations maîtresse stochastiques plus générales dérivées heuristiquement du Théorème 2.2 pour le cas discret, où les imperfections de mesure sont caractérisées par des modèles stochastiques classiques.

La théorie de filtrage quantique fournit une base de l'inférence statique inspirée, par exemple, par les systèmes optiques quantiques. Ces systèmes sont décrits par des équations différentielles stochastiques quantiques en temps continu. Ces équations maîtresses stochastiques incluent des effets de retour de mesure sur l'état quantique. À ces équations stochastiques sont attachés des filtres dits quantiques fournissant les estimations des états quantiques à partir des mesures en temps réels. La robustesse et la convergence de telles estimations ont été étudiées dans de nombreux articles. Par exemple, les conditions suffisantes relatives aux questions d'observabilité sont données dans [101] et [100]. Pour autant que nous sachions, il n'existe pas encore de conditions nécessaires et suffisantes de convergence générales et vérifiables. Dans ce chapitre, nous généralisons un résultat de stabilité pour les états purs (voir par exemple [33]) aux états quantique mixtes arbitraires. Plus précisément, nous montrons que la fidélité entre l'état quantique (qui pourrait être un état mixte) et l'état de son filtre quantique associé est une sous-martingale : cela signifie qu'en moyenne l'état estimé tend à se rapprocher de l'état du système. Cependant, ceci n'implique pas sa convergence asymptotique en fonction du temps. Pour prouver ces résultats de convergence, l'analyse plus spécifique en fonction de la structure précise des opérateurs Hamiltonien, Lindbladien et des opérateurs de mesure définissants le modèle du système est nécessaire. Ce travail peut être vu comme une extension du cas continu du Théorème 2.1 (c.f. la sous-section 1.2.4 pour plus de détails).

Ce chapitre est organisé come suit. Dans la Section 5.1, nous présentons les équations maîtresses stochastiques non linéaires et les filtres quantiques associés et nous énonçons le résultat principal dans le Théorème 5.1. La Section 5.2 est consacrée à la démonstration de ce résultat : d'une part, nous considérons une approximation par des équations stochastiques maîtresses conduites par des processus de Poisson (approximation de la diffusion) ; d'autre part, nous prouvons la propriété sous-martingale par une discrétisation en temps. Dans la section finale, nous proposons quelques extensions possibles de ce résultat. Ce travail a été publié dans [2].

In this part, we consider open quantum systems in continuous-time which are simultaneously in interaction with an environment and undergo some measurement. Such systems are usually described by stochastic differential equations. There are two possibilities for this description: either a stochastic Schrödinger equation (SSE) [11, 26, 8] when the quantum state remains pure and the jumps due to the environment and measurement are fully included; or a stochastic master equation (SME) [9, 11] for arbitrary mixed states when only the jumps due to the measurement are included . Moreover, these two stochastic differential equations can be described by two kinds of noises: the Poisson processes appear as the noises for the jump cases; the Wiener processes appear as the noises for the diffusive cases. More general SMEs or SSEs are simultaneously driven by both jump and diffusion terms [10].

In this chapter, we consider the most usual case of SMEs driven only by a Wiener process and we show that the associated quantum filters are stable. In the next chapter, we will present more general stochastic master equations derived heuristically from the discrete-time Theorem 2.2, where measurement imperfections are characterized by classical stochastic models.

The quantum filtering theory provides a foundation of statistical inference inspired in e.g., quantum optical systems. These systems are described by continuous-time quantum stochastic differential equations. These stochastic master equations include the measurements back-actions on the quantum-state. To these stochastic master equations are attached so-called quantum filters providing, from the real-time measurements, estimations of the quantum states. Robustness and convergence of such estimation process has been investigated in many papers. For example, sufficient convergence conditions, related to observability issues, are given in [101] and [100]. As far as we know, general and verifiable necessary and sufficient convergence conditions do not exist yet. In this chapter, we generalize a stability result for pure states (see e.g., [33]) to arbitrary mixed quantum states. More precisely, we prove that the fidelity between the quantum state (that could be a mixed state) and its associated quantum filter state is a submartingale: this means that in average, the estimated state tends to be closer to the system state. This does not however imply its asymptotic convergence for large times. To prove such convergence results, more specific analysis depending on the precise structure of the Hamiltonian, Lindbladian and measurement operators defining the system model is required. This work can be seen as an extension to the continuous-time case of Theorem 2.1 (see more details in Subsection 1.2.4).

This chapter is organized as follows. In Section 5.1, we introduce the non-linear stochastic master equations, its associated quantum filter and we state the main result in Theorem 5.1. Section 5.2 is devoted to the proof of this result: firstly, we consider an approximation via stochastic master equations driven by Poisson processes (diffusion approximation); secondly, we prove the submartingale property via a time discretization. In final section, we suggest some possible extensions of this result. This work was appeared in [2].

5.1 Main result

We consider quantum systems of finite-dimensions $1 < d < \infty$. The state space of such a system is given by the set of density matrices

$$\mathcal{D} := \{\rho \in \mathbb{C}^{d\times d} | \quad \rho = \rho^\dagger, \quad \mathrm{Tr}\,(\rho) = 1, \quad \rho \geq 0\}.$$

Formally, we consider the evolution of the real state $\rho \in \mathcal{D}$ which is described by the following stochastic master equation (cf., [16, 20, 103]),

$$d\rho_t = \mathcal{L}(\rho_t)\,dt + \Lambda(L)\rho_t\;dW_t\;, \tag{5.1}$$

where we recall the notations used in above:

- the superoperator $\mathcal{L}$ is called Lindblad operator which has the following form,

  $$\mathcal{L}(\rho) := -i[H,\rho] - \frac{1}{2}\{L^\dagger L, \rho\} + L\rho L^\dagger,$$

  where $H = H^\dagger$ is a Hermitian operator which describes the action of external forces on the system . The notations $[A, B]$ and $\{A, B\}$ refer respectively to $AB - BA$ and $AB + BA$;

- dW_t is the Wiener process which is the following innovation
$$dW_t = dy_t - \operatorname{Tr}\left((L + L^\dagger)\,\rho_t\right)\,dt\,, \tag{5.2}$$
where y_t is a continuous semi-martingale with quadratic variation $\langle y, y\rangle_t = t$ (which is the observation process obtained from the system) and L is an arbitrary matrix which determines the measurement process (typically the coupling to the probe field for quantum optic systems) ;

- the superoperator Λ is defined by
$$\Lambda(c)\rho := c\rho + \rho c^\dagger - \operatorname{Tr}\left((c + c^\dagger)\rho\right)\rho.$$

All the developments remain valid, when H and L are deterministic time-varying matrices. For the sake of clarity, we do not recall below such possible time dependence.

The evolution of the quantum filter of state $\rho_t^e \in \mathcal{D}$ is described by the following stochastic master equation which depends on the time-continuous measurement y_t depending on the true quantum state ρ_t via (5.2) (see e.g., [7]):
$$d\rho_t^e = \mathcal{L}(\rho_t^e)\,dt + \Lambda(L)\rho_t^e\Big(dy_t - \operatorname{Tr}\left((L + L^\dagger)\rho_t^e\right)dt\Big). \tag{5.3}$$

Replacing dy_t by its value given in (5.2), we obtain
$$d\rho_t^e = \mathcal{L}(\rho_t^e)\,dt + \Lambda(L)\rho_t^e\,dW_t + \Lambda(L)\rho_t^e\Big(\operatorname{Tr}\left((L + L^\dagger)\rho_t\right) - \operatorname{Tr}\left((L + L^\dagger)\rho_t^e\right)\Big)dt.$$

A usual measurement of the difference between two quantum states ρ_1 and ρ_2 is given by the fidelity, a real number between 0 and 1. More precisely, the fidelity between ρ_1 and ρ_2 in $\mathcal{D}$ is given by (see [75, Chapter 9] for more details)
$$F(\rho_1, \rho_2) = \operatorname{Tr}^2\left(\sqrt{\sqrt{\rho_1}\rho_2\sqrt{\rho_1}}\right). \tag{5.4}$$

Here $F(\rho_1, \rho_2) = 1$ means $\rho_1 = \rho_2$, and $F(\rho_1, \rho_2) = 0$ means that the support of ρ_1 and ρ_2 are orthogonal. When at least one of the states ρ_1 or ρ_2 is pure (i.e., orthogonal projector of rank one), $F(\rho_1, \rho_2)$ coincides with their inner product $\operatorname{Tr}(\rho_1\rho_2)$. It is well-known that stochastic master Equations (5.1) and (5.3) leave the domain $\mathcal{D}$ positively invariant (see Lemma 3.2 and Proposition 3.3 in [71]). Therefore, the expression of fidelity given by (5.4) is well-defined. Hence, we can use Itô rules to give the following numerical approximations of SMEs (5.1) and (5.3) which imply then directly, as soon as ρ_0 and ρ_0^e belong to $\mathcal{D}$, that ρ_t and ρ_t^e remain in $\mathcal{D}$ for all $t \geq 0$.
$$\rho_{t+dt} = \frac{\left(\mathbb{1} - iH\,dt - \frac{1}{2}L^\dagger L\,dt + L\,dy_t\right)\ \rho_t\ \left(\mathbb{1} - iH\,dt - \frac{1}{2}L^\dagger L\,dt + L\,dy_t\right)^\dagger}{\operatorname{Tr}\left(\left(\mathbb{1} - iH\,dt - \frac{1}{2}L^\dagger L\,dt + L\,dy_t\right)\ \rho_t\ \left(\mathbb{1} - iH\,dt - \frac{1}{2}L^\dagger L\,dt + L\,dy_t\right)^\dagger\right)} \tag{5.5}$$

and

$$\rho^e_{t+dt} = \frac{\left(\mathbb{1} - iH\,dt - \frac{1}{2}L^\dagger L\,dt + L\,dy_t\right)\ \rho^e_t\ \left(\mathbb{1} - iH\,dt - \frac{1}{2}L^\dagger L\,dt + L\,dy_t\right)^\dagger}{\mathrm{Tr}\left(\left(\mathbb{1} - iH\,dt - \frac{1}{2}L^\dagger L\,dt + L\,dy_t\right)\ \rho^e_t\ \left(\mathbb{1} - iH\,dt - \frac{1}{2}L^\dagger L\,dt + L\,dy_t\right)^\dagger\right)}, \tag{5.6}$$

where $dy_t = \mathrm{Tr}\left((L + L^\dagger)\,\rho_t\right)\,dt + dW_t$.

We are now in position to state the main result of this chapter.

Theorem 5.1. *Consider the Markov processes (ρ_t, ρ^e_t) satisfying stochastic master Equations (5.1) and (5.3) respectively with ρ_0, ρ^e_0 in $\mathcal{D}$. Then the fidelity $F(\rho_t, \rho^e_t)$, defined in Equation (5.4), is a submartingale, i.e., $\mathbb{E}\left[F(\rho_t, \rho^e_t)|(\rho_s, \rho^e_s)\right] \geq F(\rho_s, \rho^e_s)$, for all $t \geq s$.*

We recall that the above theorem generalizes the results of [33] to arbitrary purity of the real states and quantum filters. If ρ_0 is pure, then ρ_t remains pure for all $t > 0$. In this case, $F(\rho_t, \rho^e_t)$ coincides with $\mathrm{Tr}\left(\rho_t\rho^e_t\right)$. It is proved in [33] that this Frobenius inner product is a submartingale for any initial value of ρ^e_t: $\frac{d}{dt}\,\mathbb{E}\left[\mathrm{Tr}\left(\rho_t\rho^e_t\right)\right] \geq 0$. The main idea of the proof in [33] consists in using Itô's formula to reduce the theorem to showing that $\mathbb{E}\left[\mathrm{Tr}\left(d\rho_t\rho^e_t + \rho_t d\rho^e_t + d\rho_t d\rho^e_t\right)\right] \geq 0$, and then using the shift invariance of the operator L in the dynamics (5.1) and (5.3) and choosing an appropriate value.
In the absence of any information on the purity of the real states and the quantum filter, the fidelity is given by (5.4), and the application of Itô's formula for the above expression becomes much more involved. In particular, the calculation of the cross derivatives was so complicated that it became hopeless to proceed this way. As the proof presented in the next section shows, we had to choose an undirect way to approach the theorem which allowed us to avoid the heavy calculations based on second order derivatives of F.

5.2 Proof of Theorem 5.1

We proceed in two steps.

- In the first step, we describe briefly how we obtain stochastic master Equations (5.1) and (5.3) as the limits of the stochastic master equations with Poisson processes using the diffusive limits inspired from the physical balanced Homodyne detection model [9, 109].

- In the second step, we show that the fidelity between the real state and the quantum filter which are the solutions of stochastic master equations with Poisson processes, is a submartingale.

Step 1. Take $\varsigma > 0$ a large real number and consider the evolution of the quantum state ρ^ς_t described by the following stochastic master equation derived from balanced homodyne detection scheme (see Section 6.4 of [26] or [9], [109] for more physical details):

$$\begin{aligned} d\rho^\varsigma_t = &- i[H, \rho^\varsigma_t]\,dt - \tfrac{1}{4}\Lambda^\varsigma(L^\dagger L)\rho^\varsigma_t\,dt + \Upsilon^\varsigma(L)\rho^\varsigma_t\,dN_1 \\ &- \tfrac{1}{4}\Lambda^{-\varsigma}(L^\dagger L)\rho^\varsigma_t\,dt + \Upsilon^{-\varsigma}(L)\rho^\varsigma_t\,dN_2\,, \end{aligned} \tag{5.7}$$

where the superoperators Υ^ς is defined as follows

$$\Upsilon^\varsigma(c)\rho := \frac{(c+\varsigma)\rho(c^\dagger+\varsigma)}{\mathrm{Tr}\left((c+\varsigma)\rho(c^\dagger+\varsigma)\right)} - \rho,$$

and the superoperator Λ^ς is defined by

$$\Lambda^\varsigma(c)\rho := (c+\varsigma)\rho + \rho(c^\dagger+\varsigma) - \mathrm{Tr}\left((c+c^\dagger+2\varsigma)\rho\right)\rho.$$

The superoperators $\Lambda^{-\varsigma}$ and $\Upsilon^{-\varsigma}$ are just obtained with replacing ς by $-\varsigma$ in the expressions given in above.

The two processes dN_1 and dN_2 are defined by

$$dN_1 := N_1(t+dt) - N_1(t) \quad \text{and} \quad dN_2 := N_2(t+dt) - N_2(t),$$

where N_1 and N_2 are two Poisson processes. dN_1 and dN_2 take value 1 with probabilities $\frac{1}{2}\mathrm{Tr}\left((L^\dagger+\varsigma)(L+\varsigma)\rho_t^\varsigma\right)dt$ and $\frac{1}{2}\mathrm{Tr}\left((L^\dagger-\varsigma)(L-\varsigma)\rho_t^\varsigma\right)dt$, respectively, and take value 0 with the complementary probabilities.

Similarly, the following stochastic master equation describes the infinitesimal evolution of the associated quantum filter of state $\rho_t^{e,\varsigma}$ (see [7]):

$$\begin{aligned} d\rho_t^{e,\varsigma} = &-i[H,\rho_t^{e,\varsigma}]\,dt - \tfrac{1}{4}\Lambda^\varsigma(L^\dagger L)\rho_t^{e,\varsigma}\,dt + \Upsilon^\varsigma(L)\rho_t^{e,\varsigma}dN_1 \\ &- \tfrac{1}{4}\Lambda^{-\varsigma}(L^\dagger L)\rho_t^{e,\varsigma}\,dt + \Upsilon^{-\varsigma}(L)\rho_t^{e,\varsigma}dN_2. \end{aligned} \tag{5.8}$$

The following diffusive limit is obtained by the central limit theorem when ς tends to $+\infty$ for the semi-martingale processes applied to dN_q, for $q=1,2$, (see [56] or [46] for more details)

$$dN_q \xrightarrow{\text{law}} \langle \tfrac{dN_q}{dt}\rangle\,dt + \sqrt{\langle \tfrac{dN_q}{dt}\rangle}\,dW_q\,, \tag{5.9}$$

where the notation $\langle A\rangle$ refers to the mean value of A. Here

$$\langle dN_1\rangle = \frac{1}{2}\mathrm{Tr}\left((L^\dagger+\varsigma)(L+\varsigma)\rho_t^\varsigma\right)dt \quad \text{and} \quad \langle dN_2\rangle = \frac{1}{2}\mathrm{Tr}\left((L^\dagger-\varsigma)(L-\varsigma)\rho_t^\varsigma\right)dt,$$

and dW_1 and dW_2 are two independent Wiener processes, the convergence in (5.9) is in law.

Stochastic master Equations (5.1) and (5.3) are obtained by replacing the processes dN_q for $q\in\{1,2\}$ by their limits given in (5.9), in stochastic master Equations (5.7) and (5.8), taking the limit when ς goes to $+\infty$, and keeping only the lowest ordered terms in ς^{-1}. This is usually called diffusion approximation (see e.g., [29]).

Notice that dW appearing in stochastic master Equations (5.1) and (5.3) is given in terms of its independent constituents by

$$dW = \sqrt{\frac{1}{2}}\,(dW_1 + dW_2),$$

and is thus itself a standard Wiener process.

The following theorem from [78] justifies the diffusion approximation described above.

Theorem 5.2 (Pellegrini-Petruccione [78]). *The solutions of the stochastic master Equations (5.7) and (5.8) converge in law, when $\varsigma \to +\infty$, to the solutions of stochastic master Equations (5.1) and (5.3), respectively.*

Step 2. We now prove that the fidelity between two arbitrary solutions of stochastic master Equations (5.7) and (5.8) is a submartingale.

Proposition 5.1. *Consider the Markov processes $(\rho^\varsigma, \rho^{e,\varsigma})$ which satisfy stochastic master Equations (5.7) and (5.8). Then the fidelity defined in Equation (5.4) is a submartingale, i.e.,* for all $t \geq s$,

$$\mathbb{E}\left[F(\rho_t^\varsigma, \rho_t^{e,\varsigma})|(\rho_s^\varsigma, \rho_s^{e,\varsigma})\right] \geq F(\rho_s^\varsigma, \rho_s^{e,\varsigma}).$$

Proof. We consider approximations of the time-continuous Markov processes (5.7) and (5.8) by discrete-time Markov processes ξ_k and ξ_k^e:

$$\xi_{k+1} = \frac{M_{\mu_k}\xi_k M_{\mu_k}^\dagger}{\mathrm{Tr}\left(M_{\mu_k}\xi_k M_{\mu_k}^\dagger\right)} \quad \text{and} \quad \xi_{k+1}^e = \frac{M_{\mu_k}\xi_k^e M_{\mu_k}^\dagger}{\mathrm{Tr}\left(M_{\mu_k}\xi_k^e M_{\mu_k}^\dagger\right)}, \tag{5.10}$$

where

- $k \in \{0, \dots, n\}$ for a fixed large n;
- $\xi_0 = \rho_s^\varsigma$ and $\xi_0^e = \rho_s^{e,\varsigma}$;
- μ_k is a random variable taking values $\mu \in \{0, 1, 2\}$ with probability $P_{\mu,k} = \mathrm{Tr}\left(M_\mu \xi_k M_\mu^\dagger\right)$;
- The operators M_0, M_1 and M_2 are defined as follows

$$M_0 := 1 - \tfrac{1}{4}(L^\dagger + \varsigma)(L + \varsigma)\epsilon_n - \tfrac{1}{4}(L^\dagger - \varsigma)(L - \varsigma)\epsilon_n - iH\epsilon_n,$$

$$M_1 := (L + \varsigma)\sqrt{\frac{1}{2}\epsilon_n}, \quad \text{and} \quad M_2 := (L - \varsigma)\sqrt{\frac{1}{2}\epsilon_n},$$

with $\epsilon_n = \frac{t-s}{n}$.

In the following lemma, we show that ξ_n and ξ_n^e correspond to the Euler-Maruyama time discretization. Since (5.7) and (5.8) depend smoothly on ρ_t^ς and $\rho_t^{e,\varsigma}$, ξ_n and ξ_n^e converge in law towards ρ_t^ς and $\rho_t^{e,\varsigma}$, when $n \to +\infty$.

Lemma 5.1. *The processes ξ_k and ξ_k^e correspond up to second order terms in ϵ_n, to the Euler-Maruyama discretization scheme of (5.7) and (5.8) on $[s, t]$.*

Proof. we regard the three following possible cases which may happen according to the different values of μ_k . In each case, we show that ξ_k and ξ_k^e for $k \in \{0, \dots, n\}$ are the numerical solutions of the dynamics (5.7) and (5.8) respectively, with the following partition $s \leq s + \epsilon_n \leq \dots \leq s + (n-1)\epsilon_n \leq t$, where the uniform step length ϵ_n is $\frac{t-s}{n}$.

Case 1. We first consider the case where $\mu_k = 0$ which happens with probability $P_{0,k} = \mathrm{Tr}\left(M_0 \xi_k M_0^\dagger\right)$. Note that

$$M_0\xi_k M_0^\dagger = \xi_k - \tfrac{1}{4}\{(L^\dagger+\varsigma)(L+\varsigma),\xi_k\}\,\epsilon_n - \tfrac{1}{4}\{(L^\dagger-\varsigma)(L-\varsigma),\xi_k\}\,\epsilon_n - i[H,\xi_k]\,\epsilon_n + \mathcal{O}(\epsilon_n^2).$$

Therefore

$$\mathrm{Tr}\left(M_0\xi_k M_0^\dagger\right) = 1 - \frac{1}{2}\mathrm{Tr}\left((L^\dagger+\varsigma)(L+\varsigma)\xi_k\right)\,\epsilon_n - \frac{1}{2}\mathrm{Tr}\left((L^\dagger-\varsigma)(L-\varsigma)\xi_k\right)\,\epsilon_n + \mathcal{O}((\epsilon_n)^2)$$

and

$$\left(\mathrm{Tr}\left(M_0\xi_k M_0^\dagger\right)\right)^{-1} \approx 1 + \frac{1}{2}\mathrm{Tr}\left((L^\dagger+\varsigma)(L+\varsigma)\xi_k\right)\,\epsilon_n + \frac{1}{2}\mathrm{Tr}\left((L^\dagger-\varsigma)(L-\varsigma)\xi_k\right)\,\epsilon_n + \mathcal{O}((\epsilon_n)^2).$$

Therefore, we find the following dynamics

$$\begin{aligned}\xi_{k+1} \approx \xi_k &- \tfrac{1}{4}\{(L^\dagger+\varsigma)(L+\varsigma),\xi_k\}\,\epsilon_n - \tfrac{1}{4}\{(L^\dagger-\varsigma)(L-\varsigma),\xi_k\}\,\epsilon_n \\ &+ \frac{1}{2}\mathrm{Tr}\left((L^\dagger+\varsigma)(L+\varsigma)\xi_k\right)\,\xi_k\,\epsilon_n + \frac{1}{2}\mathrm{Tr}\left((L^\dagger-\varsigma)(L-\varsigma)\xi_k\right)\,\xi_k\,\epsilon_n + \mathcal{O}(\epsilon_n^2).\end{aligned}$$

This can be also written as follows

$$\xi_{k+1} - \xi_k \approx -\tfrac{1}{4}\Lambda^{\varsigma}(L^\dagger L)\xi_k\,\epsilon_n - \tfrac{1}{4}\Lambda^{-\varsigma}(L^\dagger L)\xi_k\,\epsilon_n + \mathcal{O}(\epsilon_n^2). \tag{5.11}$$

Obviously, this dynamics in the first order of ϵ_n is equivalent to the dynamics of the numerical solution of stochastic master Equation (5.7) with the partition $s \leq s+\epsilon_n \leq \ldots \leq s+(n-1)\epsilon_n \leq t$, when

$$N_1(s+(k+1)\epsilon_n) - N_1(s+k\epsilon_n) = 0 \quad \text{and} \quad N_2(s+(k+1)\epsilon_n) - N_2(s+k\epsilon_n) = 0,$$

which happens with probability

$$\left(1 - \frac{1}{2}\mathrm{Tr}\left((L+\varsigma)(L^\dagger+\varsigma)\,\xi_k\right)\,\epsilon_n\right)\left(1 - \frac{1}{2}\mathrm{Tr}\left((L-\varsigma)(L^\dagger-\varsigma)\,\xi_k\right)\,\epsilon_n\right).$$

This probability, in the first order of ϵ_n is equal to $\mathrm{Tr}\left(M_0\xi_k M_0^\dagger\right)$.

Case 2. The second case corresponds to $\mu_k = 1$ which happens with probability $\mathrm{Tr}\left(M_1\xi_k M_1^\dagger\right)$. We find the following dynamics

$$\xi_{k+1} = \frac{(L+\varsigma)\xi_k(L^\dagger+\varsigma)}{\mathrm{Tr}\left((L+\varsigma)\xi_k(L^\dagger+\varsigma)\right)} = \Upsilon^{\varsigma}(L)\,\xi_k + \xi_k.$$

We observe that the numerical solution of stochastic master Equation (5.7) follows also the same dynamics when

$$N_1(s+(k+1)\epsilon_n) - N_1(s+k\epsilon_n) = 1 \quad \text{and} \quad N_2(s+(k+1)\epsilon_n) - N_2(s+k\epsilon_n) = 0,$$

which happens with probability

$$\big(\frac{1}{2}\mathrm{Tr}\left((L+\varsigma)(L^\dagger+\varsigma)\,\xi_k\right)\,\epsilon_n\big)\big(1-\frac{1}{2}\mathrm{Tr}\left((L-\varsigma)(L^\dagger-\varsigma)\,\xi_k\right)\,\epsilon_n\big).$$

This is equal to $\mathrm{Tr}\left(M_1\xi_k M_1^\dagger\right)$, in the first order of ϵ_n.

Case 3. Now we consider the last case $\mu_k = 2$ which happens with probability $\mathrm{Tr}\left(M_2\xi_k M_2^\dagger\right)$. Therefore, we have

$$\xi_{k+1} = \frac{(L-\varsigma)\xi_k(L^\dagger-\varsigma)}{\mathrm{Tr}\left((L-\varsigma)\xi_k(L^\dagger-\varsigma)\right)} = \Upsilon^{-\varsigma}(L)\xi_k + \xi_k.$$

This can be also written by stochastic master Equation (5.7) with taking ξ_k as the numerical solution and

$$N_1(s+(k+1)\epsilon_n) - N_1(s+k\epsilon_n) = 0 \quad \text{and} \quad N_2(s+(k+1)\epsilon_n) - N_2(s+k\epsilon_n) = 1,$$

which happens with probability

$$\big(1-\frac{1}{2}\mathrm{Tr}\left((L+\varsigma)(L^\dagger+\varsigma)\,\xi_k\right)\,\epsilon_n\big)\big(\frac{1}{2}\mathrm{Tr}\left((L-\varsigma)(L^\dagger-\varsigma)\,\xi_k\right)\,\epsilon_n\big),$$

where in the first order of ϵ_n, this probability is equal to $\mathrm{Tr}\left(M_2\xi_k M_2^\dagger\right)$.

Remark that if we neglect the terms in the order of ϵ_n^2, the probability of $N_1(s+(k+1)\epsilon_n) - N_1(s+k\epsilon_n) = 1$ and $N_2(s+(k+1)\epsilon_n) - N_2(s+k\epsilon_n) = 1$ is negligible. Now it is clear that ξ_k and similarly ξ_k^e are respectively the numerical solutions of stochastic master Equations (5.7) and (5.8) obtained by Euler-Maruyama method. As the right hand side of stochastic master Equations (5.7) and (5.8) are smooth with respect to ρ^ς and $\rho^{e,\varsigma}$, we can use the result of [45, Theorem 1] to conclude the convergence in law of ξ_n and ξ_n^e to ρ_t^ς and $\rho_t^{e,\varsigma}$ for large n. □

Now we notice that

$$M_0^\dagger M_0 + M_1^\dagger M_1 + M_2^\dagger M_2 = \mathbb{1} + \mathcal{O}(\epsilon_n^2) =: A,$$

Take $\widetilde{M_r} := (\sqrt{A})^{-1}M_r$ for $r = 0,1,2$ which satisfy necessarily

$$\widetilde{M_0}^\dagger\widetilde{M_0} + \widetilde{M_1}^\dagger\widetilde{M_1} + \widetilde{M_2}^\dagger\widetilde{M_2} = \mathbb{1}. \tag{5.12}$$

We define now the following Markov processes υ_k and υ_k^e by

$$\upsilon_{k+1} = \frac{\widetilde{M_{\mu_k}}\upsilon_k\widetilde{M_{\mu_k}}^\dagger}{\mathrm{Tr}\left(\widetilde{M_{\mu_k}}\upsilon_k\widetilde{M_{\mu_k}}^\dagger\right)} \tag{5.13}$$

and

$$\upsilon_{k+1}^e = \frac{\widetilde{M_{\mu_k}}\upsilon_k^e\widetilde{M_{\mu_k}}^\dagger}{\mathrm{Tr}\left(\widetilde{M_{\mu_k}}\upsilon_k^e\widetilde{M_{\mu_k}}^\dagger\right)}, \tag{5.14}$$

where

- $k \in \{0, \dots, n\}$ for a fixed large n;
- $\upsilon_0 = \xi_0$ and $\upsilon_0^e = \xi_0^e$;
- μ_k is a random variable taking values $\mu \in \{0, 1, 2\}$ with probability $P_{\mu,k} = \mathrm{Tr}\left(\widetilde{M}_\mu \upsilon_k \widetilde{M}_\mu^\dagger\right)$.

Clearly υ_k and υ_k^e can be seen as the numerical solutions of stochastic master Equations (5.7) and (5.8), since $(\sqrt{A})^{-1} = \mathbb{1} - \mathcal{O}(\epsilon_n^2)$. Therefore in the first order of ϵ_n, the solutions ξ_k and ξ_k^e are equal to υ_k and υ_k^e, respectively. But the advantage of using υ_k and υ_k^e instead of ξ_k and ξ_k^e is that the operators $\widetilde{M}_r$ are the Kraus operators, since they satisfy Equality (5.12). Thus we can apply Theorem 2.1, which proves that $F(\upsilon_k, \upsilon_k^e)$ is a submartingale. Thus we have

$$\mathbb{E}\left[F(\upsilon_n, \upsilon_n^e) \mid \upsilon_0, \upsilon_0^e\right] \geq F(\upsilon_0, \upsilon_0^e) = F(\rho_s^\varsigma, \rho_s^{e,\varsigma}).$$

Therefore by Lemma 5.1, we have necessarily

$$\mathbb{E}\left[F(\rho_t^\varsigma, \rho_t^{e,\varsigma}) | \rho_s^\varsigma, \rho_s^{e,\varsigma}\right] \geq F(\rho_s^\varsigma, \rho_s^{e,\varsigma}),$$

for all $t \geq s$, since we have (convergence in law) $\rho_t^\varsigma = \lim_{n \longrightarrow \infty} \upsilon_n$, $\rho_t^{e,\varsigma} = \lim_{n \longrightarrow \infty} \upsilon_n^e$, $\upsilon_0 = \rho_s^\varsigma$ and $\upsilon_0^e = \rho_s^{e,\varsigma}$. $\square$

We now apply Theorem 5.2 and we use the fact that the function F is bounded by one and is continuous with respect to ρ and ρ^e to get

$$\mathbb{E}\left[F(\rho_t, \rho_t^e) | (\rho_s, \rho_s^e)\right] \geq F(\rho_s, \rho_s^e),$$

for all $t \geq s$, which ends the proof of Theorem 5.1.

5.3 Numerical test

In this section, we test the result of Theorem 5.1 through numerical simulations. Considering the two-level system of [103], we take the following Hamiltonian H and measurement operators L:

$$H = \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad L = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The simulations of figure 5.1 illustrate the fidelity for 500 random trajectories starting at

$$\rho_0 = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{pmatrix} \quad \text{and} \quad \rho_0^e = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{pmatrix}.$$

In particular, we note that both initial states are mixed ones. As it can be seen, the average fidelity is monotonically increasing. Here the fidelity converges to one indicating the convergence of the filter towards the physical state. An interesting direction is to characterize the situations where this convergence is ensured.

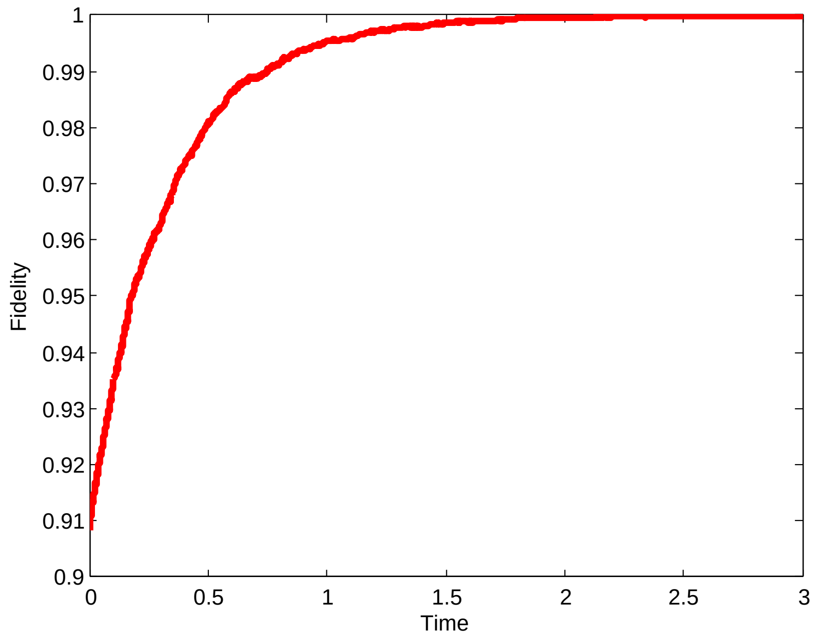

Figure 5.1: The average fidelity between the Markov processes ρ and ρ^e, over 500 realizations, time t from 0 to $T = 3$ with discretization time step $dt = 10^{-4}$.

In order to simulate Equations (5.1) and (5.3), we have considered the alternative formulations (5.5) and (5.6) and the resulting discretization scheme ($k \in \mathbb{N}$ and time step $0 < dt \ll 1$)

$$\rho_{(k+1)dt} = \frac{\mathcal{M}_k \ \rho_{(kdt)} \ \mathcal{M}_k^\dagger}{\mathrm{Tr}\left(\mathcal{M}_k \rho_{(kdt)} \mathcal{M}_k^\dagger\right)}, \quad \rho^e_{(k+1)dt} = \frac{\mathcal{M}_k \rho^e_{(kdt)} \mathcal{M}_k^\dagger}{\mathrm{Tr}\left(\mathcal{M}_k \rho^e_{(kdt)} \mathcal{M}_k^\dagger\right)},$$

where $\mathcal{M}_k = \mathbb{1} - iH\,dt - \frac{1}{2}L^\dagger L dt + L dy_{(kdt)}$ and $dy_{(kdt)} = \mathrm{Tr}\left((L + L^\dagger)\,\rho_{(kdt)}\right)\,dt + dW_{(kdt)}$. For each k, the Wiener increment $dW_{(kdt)}$ is a centered Gaussian random variable of standard deviation $\sqrt{dt}$. As it said before, the major interest of such discretization is to guaranty that, if $\rho_0, \rho_0^e \in \mathcal{D}$, then ρ_k and ρ_k^e also remain in $\mathcal{D}$ for any $k \geq 0$.

5.4 Conclusion

The fact that the fidelity between the real quantum state and the quantum filter state increases in average should remain valid for more general stochastic master equations where other Lindbald terms are added to $\mathcal{L}(\rho)$ appearing in (5.1). We briefly discuss the necessary change here. In this case, the dynamics (5.1) and (5.3) become

$$d\rho_t = -i[H, \rho_t]\,dt + \sum_{\widetilde{\nu}=1}^{\widetilde{p}} \widetilde{\mathcal{L}}_{\widetilde{\nu}}(\rho_t)\,dt + \sum_{\nu=1}^{p} \mathcal{L}_\nu(\rho_t)\,dt + \sum_{\nu=1}^{p} \Lambda_\nu(L_\nu)\rho_t\,dW_t^\nu,$$

and

$$d\rho_t^e = -i[H,\rho_t^e]\,dt + \sum_{\widetilde{\nu}=1}^{\widetilde{p}} \widetilde{\mathcal{L}}_{\widetilde{\nu}}(\rho_t^e)\,dt + \sum_{\nu=1}^{p} \mathcal{L}_\nu(\rho_t^e)\,dt + \sum_{\nu=1}^{p} \Lambda_\nu(L_\nu)\rho_t^e \Big(dy_t^\nu - \operatorname{Tr}\left((L_\nu + L_\nu^\dagger)\rho_t^e\right) dt \Big),$$

where dW_t^ν are independent Wiener processes,

$$\mathcal{L}_\nu(\rho) := -\frac{1}{2}\{L_\nu^\dagger L_\nu, \rho\} + L_\nu \rho L_\nu^\dagger,$$

$$\widetilde{\mathcal{L}}_{\widetilde{\nu}}(\rho) := -\frac{1}{2}\{\widetilde{L}_{\widetilde{\nu}}^\dagger \widetilde{L}_{\widetilde{\nu}}, \rho\} + \widetilde{L}_{\widetilde{\nu}} \rho \widetilde{L}_{\widetilde{\nu}}^\dagger,$$

and $\Lambda_\nu(c)\rho := c\rho + \rho c^\dagger - \operatorname{Tr}\left((c+c^\dagger)\rho\right)\rho$. Here $p, \widetilde{p} \geq 1$, $(\widetilde{L}_{\widetilde{\nu}})_{1\leq \widetilde{\nu} \leq \widetilde{p}}$ and $(L_\nu)_{1\leq\nu\leq p}$ are arbitrary operators. The special case considered here corresponds to $p = 1$ and $\widetilde{p} = 1$ with $L_1 = L$ and $\widetilde{L}_1 = 0$. For this general case, the proof of Theorem 5.1 should follow the same lines: first step still relies on Theorem 5.2; second step relies now on [82, Theorem 2]. The formulations analogue to (5.5) and (5.6) read then

$$\rho_{t+dt} = \frac{dM_t\ \rho_t\ dM_t^\dagger + \sum_{\widetilde{\nu}=1}^{\widetilde{p}} \widetilde{L}_{\widetilde{\nu}} \rho_t \widetilde{L}_{\widetilde{\nu}}^\dagger\, dt}{\operatorname{Tr}\left(dM_t\ \rho_t\ dM_t^\dagger + \sum_{\widetilde{\nu}=1}^{\widetilde{p}} \widetilde{L}_{\widetilde{\nu}} \rho_t \widetilde{L}_{\widetilde{\nu}}^\dagger\, dt\right)},$$

and

$$\rho_{t+dt}^e = \frac{dM_t\ \rho_t^e\ dM_t^\dagger + \sum_{\widetilde{\nu}=1}^{\widetilde{p}} \widetilde{L}_{\widetilde{\nu}} \rho_t^e \widetilde{L}_{\widetilde{\nu}}^\dagger\, dt}{\operatorname{Tr}\left(dM_t\ \rho_t^e\ dM_t^\dagger + \sum_{\widetilde{\nu}=1}^{\widetilde{p}} \widetilde{L}_{\widetilde{\nu}} \rho_t^e \widetilde{L}_{\widetilde{\nu}}^\dagger\, dt\right)},$$

where denoting dy_t^ν and dM_t by

$$dy_t^\nu = dW_t^\nu + \operatorname{Tr}\left((L_\nu + L_\nu^\dagger)\rho_t\right)\, dt,$$

and

$$dM_t = \mathbb{1} - iH\,dt - \frac{1}{2}\sum_{\widetilde{\nu}=1}^{\widetilde{p}} \widetilde{L}_{\widetilde{\nu}}^\dagger \widetilde{L}_{\widetilde{\nu}}\,dt - \frac{1}{2}\sum_{\nu=1}^{p} L_\nu^\dagger L_\nu\,dt + \sum_{\nu=1}^{p} L_\nu dy_t^\nu.$$

Chapter 6

Models of continuous-time open quantum systems

Contents

Dans le chapitre précédent, nous avons étudié la stabilité du filtre quantique associé à une équation maîtresse stochastique découlant d'un processus de Wiener.

Dans ce chapitre, nous présentons des SMEs plus générales qui découlent soit de processus de Poisson multidimensionnel, soit de processus de Wiener, soit des deux à la fois. Notre contribution consiste ici à ajouter des erreurs de mesures classiques décrites par des modèles stochastiques classiques afin de construire heuristiquement les SMEs découlant par des processus de Poisson et Wiener multidimensionnels (SME (6.16)). Une telle SME est une extension naturelle de la chaîne de Markov en temps discret (2.7) au temps continu.

Ce chapitre est structuré comme suit. Les Sections 6.1 et 6.2 sont respectivement consacrées aux SMEs avec des processus de Poisson et Wiener multidimensionnels en présence d'imperfections de mesures. Dans la Section 6.3, nous considérons une SME générale découlant à la fois par des processus de Poisson et Wiener multidimensionnels : la stabilité du filtre associé est exprimée dans la Conjecture 1. Nous rappelons que les résultats obtenus dans ce chapitre sont basés sur des arguments heuristiques. La manière rigoureuse d'obtenir les SMEs génériques découlant des deux processus de Poisson et Wiener multidimensionnels en présence d'erreurs de mesure classiques ainsi qu'une preuve rigoureuse de la Conjecture 1 sont encore des travaux en cours [4]. Pour plus d'interprétations physiques sur le contenu de ce chapitre, nous-nous référons à [69], [92], [112] et [44].

In the previous chapter, we studied the stability of the quantum filter associated to a stochastic master equation driven by a Wiener process.

In this chapter, we present more general SMEs which are driven either by multidimensional Poisson processes, Wiener processes or both of them. Our contribution consists here in adding measurement errors described by classical stochastic models to design heuristically the SMEs driven by both multidimensional Poisson and Wiener processes (SME (6.16)). Such SME is a natural extension to continuous-time of discrete-time Markov chain (2.7).

This chapter is structured as follows. Sections 6.1 and 6.2 are respectively devoted to the SMEs with multidimensional Poisson and Wiener processes in presence of measurements imperfections. In Section 6.3, we consider a generic SME driven by both multidimensional Poisson and Wiener processes: the stability of the associated quantum filter is expressed through Conjecture 1. We remind that the results obtained in this chapter are based on some heuristic arguments. The rigorous way to obtain the generic SMEs given by both multidimensional Poisson and Wiener processes, in presence of classical measurement errors, and a rigorous proof of Conjecture 1 is in progress [4]. For more physical interpretations of the contents of this chapter, we refer to [69], [92], [112] and [44].

6.1 SMEs driven by multidimensional Poisson processes

In this section, we consider the SMEs with m-dimensional Poisson processes for perfect and imperfect measurements.

6.1.1 Perfect measurement

We consider a finite-dimensional quantum system (the underlying Hilbert space $\mathcal{H} = \mathbb{C}^d$ is of dimension $d > 0$) which is in interaction with its environment. This system is being measured through a continuous-in-time POVM (see more physical details in [44]). The system state is described by the density matrix $\rho \in \mathcal{D}$:

$$\mathcal{D} := \{\rho \in \mathbb{C}^{d\times d} \mid \quad \rho = \rho^\dagger, \quad \mathrm{Tr}\,(\rho) = 1, \quad \rho \geq 0\}.$$

In the aim of modeling the evolution, we consider the result of the measurement in the interval $[t, t + dt]$ (this model is the analogue of the discrete-time model, since we regard the infinitesimal evolution of the system). In this interval, we detect necessarily one of the outcomes $\mu \in \{0, 1, \ldots, m\}$.

For any $\mu \neq 0$, we define the following jump operators

$$M_\mu = \sqrt{dt}\, C_\mu, \tag{6.1}$$

where C_μ is an arbitrary operator defined in $\mathbb{C}^{d\times d}$. When the detector observes the outcome μ, the evolution is governed by the following jump

$$\rho_{t+dt} = \frac{M_\mu \rho_t M_\mu^\dagger}{\operatorname{Tr}\left(M_\mu \rho_t M_\mu^\dagger\right)} = \frac{C_\mu \rho_t {C_\mu}^\dagger}{\operatorname{Tr}\left(C_\mu \rho_t {C_\mu}^\dagger\right)},$$

which happens with probability $\operatorname{Tr}\left(C_\mu \rho_t {C_\mu}^\dagger\right) dt$.

The other situation ($\mu = 0$) corresponds to no-jump case, this has an influence on the system and can be described by the measurement operator M_0, but using the fact that the measurement operators are POVM, we have

$$\sum_{\mu=1}^{m} M_\mu^\dagger M_\mu + M_0^\dagger M_0 = \mathbb{1}.$$

Thus, in the first order of dt, the no-jump operator M_0 has necessarily the following form

$$M_0 = \mathbb{1} - \frac{1}{2}\sum_{\mu=1}^{m} {C_\mu}^\dagger C_\mu \, dt - i\, H\, dt, \tag{6.2}$$

where H describes the Hamiltonian of the system. Hence we find the following dynamics for no-jump detected

$$\begin{aligned}\rho_{t+dt} &= \frac{M_0 \rho_t M_0^\dagger}{\operatorname{Tr}\left(M_0 \rho_t M_0^\dagger\right)} \\ &= \rho_t - \sum_{\mu=1}^{m} \frac{1}{2}\left({C_\mu}^\dagger C_\mu \rho_t + \rho_t {C_\mu}^\dagger C_\mu\right) dt + \sum_{\mu=1}^{m} \operatorname{Tr}\left(C_\mu \rho_t {C_\mu}^\dagger\right)\, \rho_t \, dt - \, i[H, \rho_t]\, dt,\end{aligned}$$

which happens with probability $\operatorname{Tr}\left(M_0 \rho_t M_0^\dagger\right) = 1 - \sum_{\mu=1}^{m} \operatorname{Tr}\left(C_\mu \rho_t {C_\mu}^\dagger\right) dt$, in the first order of dt.

In order to summarize, the stochastic evolution of the density matrix ρ_t is modeled through the following trajectories, known as quantum Monte Carlo trajectories.

$$\rho_{t+dt} = \begin{cases} \frac{C_\mu \rho_t {C_\mu}^\dagger}{\operatorname{Tr}\left(C_\mu \rho_t {C_\mu}^\dagger\right)}, \quad \text{with probability} \quad \operatorname{Tr}\left(C_\mu \rho_t {C_\mu}^\dagger\right) dt, \quad \text{for any } \mu \in \{1, \ldots, m\}, \\ \rho_t - \sum_{\mu=1}^{m} \frac{1}{2}\left({C_\mu}^\dagger C_\mu \rho_t + \rho_t {C_\mu}^\dagger C_\mu\right) dt + \sum_{\mu=1}^{m} \operatorname{Tr}\left(C_\mu \rho_t {C_\mu}^\dagger\right)\, \rho_t \, dt - \, i[H, \rho_t]\, dt, \\ \qquad \text{with probability} \quad 1 - \sum_{\mu=1}^{m} \operatorname{Tr}\left(C_\mu \rho_t {C_\mu}^\dagger\right) dt. \end{cases}$$

We can combine these $m+1$ possibilities by applying m Poisson processes. In any given time interval $[t, t+dt[$, we define the Poisson process $N^\mu(t)$ and the processes

$$dN_t^\mu := N^\mu(t + dt) - N^\mu(t),$$

which takes the value one with probability $\mathrm{Tr}\left(C_\mu \rho_t {C_\mu}^\dagger\right) dt$ and value zero with $1 - \mathrm{Tr}\left(C_\mu \rho_t {C_\mu}^\dagger\right) dt$. Hence, the above dynamics can be represented through the following Itô stochastic master equation (which has been obtained by taking a conditional expectation on the actual state and the previous observations)

$$d\rho_t = \mathcal{L}(\rho_t)\,dt + \sum_{\mu=1}^{m} \Upsilon_\mu(\rho_t)\left(dN_t^\mu - \mathrm{Tr}\left(C_\mu \rho_t C_\mu^\dagger\right)\, dt\right), \tag{6.3}$$

where

- the Lindblad operator $\mathcal{L}$ is given by the following expression

$$\mathcal{L}(\rho) := -i\,[H, \rho] + \sum_{\mu=1}^{m} \mathcal{L}_\mu(\rho),$$

  with $\mathcal{L}_\mu(\rho) := C_\mu \rho C_\mu^\dagger - \frac{1}{2}\{C_\mu^\dagger C_\mu, \rho\}$;
- the superoperator Υ_μ is defined as follows

$$\Upsilon_\mu(\rho) := \frac{C_\mu \rho C_\mu^\dagger}{\mathrm{Tr}\left(C_\mu \rho C_\mu^\dagger\right)} - \rho.$$

In fact, if we consider the situation of an ensemble of such quantum systems, then we get the following average dynamics (using the fact that $\langle dN_t^\mu \rangle = \mathrm{Tr}\left(C_\mu \rho_t C_\mu^\dagger\right)\, dt$)

$$d\langle \rho_t \rangle = \mathcal{L}(\langle \rho_t \rangle)\,dt,$$

where the notation $\langle\rangle$ stands for the expectation value. This average dynamics is called Lindblad master equation.

An example of Poisson process: spontaneous emission. A simple example is a two level atom which is in interaction with a quasi resonant external optical field. The atom can also interact with the vacuum modes of the free radiation field (the environment). This induces to spontaneous jumps of atom from its excited state $|e\rangle$ to the ground state $|g\rangle$ which causes the emission of a photon in a random direction. The state of the system lives in the two-dimensional Hilbert space spanned by $|e\rangle$ and $|g\rangle$. In the interval $[t, t+dt]$ of time, either we detect a photon or none. The probability to detect a photon is proportional to the population of the excited state $\mathrm{Tr}\left(|e\rangle\langle e|\,\rho\right) = \langle e|\,\rho\,|e\rangle$. More precisely, this probability can be expressed through the following formula

$$P_1 = \Gamma\,\langle e|\,\rho\,|e\rangle\,dt.$$

Here Γ is the decay rate of the system which is equivalent to the inverse of the atomic lifetime of the excited state $|e\rangle$. The measurement operator corresponding to the jump phenomena is given by the following

$$M_1 = \sqrt{\Gamma dt}\,\sigma_-,$$

where $\sigma_- = |g\rangle\langle e|$. Once we detect a photon, the excited state will jump to the ground state. So the evolution in this case is governed through the following dynamics:

$$\rho_{t+dt} = \frac{M_1 \rho_t M_1^\dagger}{\mathrm{Tr}\left(M_1 \rho_t M_1^\dagger\right)} = \frac{\sigma_- \rho_t \sigma_-^\dagger}{\mathrm{Tr}\left(\sigma_- \rho_t \sigma_-^\dagger\right)} = |g\rangle\langle g|,$$

which happens with probability $\Gamma \mathrm{Tr}\left(\sigma_- \rho_t \sigma_-^\dagger\right) dt$.

The no-jump measurement operator has the following form

$$M_0 = \mathbb{1} - \frac{1}{2}\Gamma \sigma_-^\dagger \sigma_-\, dt - i\, H\, dt,$$

where $\sigma_-^\dagger = \sigma_+ = |e\rangle\langle g|$ and H is the Hamiltonian of the system. Thus we find the following dynamics for no-jump detected, if we neglect the second order terms in dt:

$$\begin{aligned}\rho_{t+dt} &= \frac{M_0 \rho_t M_0^\dagger}{\mathrm{Tr}\left(M_0 \rho_t M_0^\dagger\right)} \\ &= \rho_t - \tfrac{\Gamma}{2}\{\sigma_+\sigma_-, \rho_t\}\, dt + \Gamma \mathrm{Tr}\left(\sigma_- \rho_t \sigma_+\right)\rho_t\, dt - i\,[H, \rho_t]\, dt,\end{aligned}$$

which arrives with probability $\mathrm{Tr}\left(M_0 \rho_t M_0^\dagger\right) = 1 - \Gamma \mathrm{Tr}\left(\sigma_- \rho_t \sigma_+\right) dt$, in the first order of dt. We can now define a Poisson process $N(t)$ and the process $dN_t := N(t+dt) - N(t)$, which takes the value one with probability $\Gamma \mathrm{Tr}\left(\sigma_- \rho_t \sigma_+\right) dt$ and zero with the complementary probability. The following SME describes the evolution of the system:

$$d\rho_t = \mathcal{L}(\rho_t)\, dt + \left(\frac{\sigma_- \rho_t \sigma_+}{\mathrm{Tr}\left(\sigma_- \rho_t \sigma_+\right)} - \rho_t\right)\left(dN_t - \Gamma \mathrm{Tr}\left(\sigma_- \rho_t \sigma_+\right) dt\right),$$

where the Lindblad operator is given by $\mathcal{L}(\rho) = -i\,[H, \rho] + \Gamma \sigma_- \rho \sigma_+ - \frac{1}{2}\Gamma\{\sigma_+\sigma_-, \rho\}$. One easily verifies that the above SME is a special example of SME (6.3) by taking $m = 1$ and $C_1 = \sqrt{\Gamma dt}\sigma_-$.

6.1.2 Imperfect measurement

Assume that there exists the unread measurements performed by the environment and the active measurements performed by the non-ideal detectors like the situations described in Section 2.7. Set the ideal outcome μ in the set $\{0, 1, \dots, m\}$ and suppose that the real outcome μ' takes its values in the set $\{0, 1, \dots, m'\}$. The Kraus operators attached to the ideal outcome μ are denoted by M_μ which are given by (6.1) and (6.2). We use the notation η presented in Section 2.7 for the stochastic matrix characterizing the classical correlation between the events μ and μ'. We suppose that the elements of η are known and take the following values:

- for any $\mu' \neq 0$, $\eta_{\mu',0} = \bar{\eta}_{\mu'}\, dt$ and $\eta_{0,0} = 1 - \sum_{\mu'=1}^{m'} \bar{\eta}_{\mu'}\, dt$, with $\bar{\eta}_{\mu'} \geq 0$;

- for any $\mu \neq 0$, $\eta_{0,\mu} = 1 - \sum_{\mu'=1}^{m'} \eta_{\mu',\mu}$, where $0 \leq \eta_{\mu',\mu} \leq 1$ and $\eta_{0,\mu} \geq 0$.

Note that we take $\eta_{\mu',0}$ in order of dt, which is a natural choice.

We are aiming to obtain the evolution of the optimal filter which is the expected value of the current system state conditioned on the initial state and all the previous detector real outcomes. Apply now Theorem 2.2 to deduce the infinitesimal evolution of the optimal filter $\widehat{\rho}_t$ in the interval $[t, t+dt]$. We have for any $\mu' \in \{0, 1, \dots, m'\}$:

$$\widehat{\rho}_{t+dt} = \mathbb{L}_{\mu'}(\widehat{\rho}_t) = \frac{\sum_{\mu=0}^{m} \eta_{\mu',\mu} M_\mu \widehat{\rho}_t M_\mu^\dagger}{\operatorname{Tr}\left(\sum_{\mu=0}^{m} \eta_{\mu',\mu} M_\mu \widehat{\rho}_t M_\mu^\dagger\right)}, \tag{6.4}$$

which happens with probability $\operatorname{Tr}\left(\sum_{\mu=0}^{m} \eta_{\mu',\mu} M_\mu \widehat{\rho}_t M_\mu^\dagger\right)$.

Now replacing the expressions of M_μ described in (6.1) and (6.2) inside the dynamics (6.4) and keeping only the first order terms of dt, we find the following for $\mu' = 0$:

$$\begin{aligned}
\mathbb{L}_0(\widehat{\rho}_t) &= \frac{\left(1 - \sum_{\mu'=1}^{m'} \bar{\eta}_{\mu'}\, dt\right)\left(\widehat{\rho}_t - \frac{1}{2}\sum_{\mu=1}^{m}\{C_\mu^\dagger C_\mu, \widehat{\rho}_t\}\, dt - i[H, \widehat{\rho}_t]\, dt\right) + \sum_{\mu=1}^{m}\left(1 - \sum_{\mu'=1}^{m'} \eta_{\mu',\mu}\right) C_\mu \widehat{\rho}_t C_\mu^\dagger\, dt}{\operatorname{Tr}\left(\left(1 - \sum_{\mu'=1}^{m'} \bar{\eta}_{\mu'}\, dt\right)\left(\widehat{\rho}_t - \frac{1}{2}\sum_{\mu=1}^{m}\{C_\mu^\dagger C_\mu, \widehat{\rho}_t\}\, dt - i[H, \widehat{\rho}_t]\, dt\right) + \sum_{\mu=1}^{m}\left(1 - \sum_{\mu'=1}^{m'} \eta_{\mu',\mu}\right) C_\mu \widehat{\rho}_t C_\mu^\dagger\, dt\right)} \\
&= \frac{\left(1 - \sum_{\mu'=1}^{m'} \bar{\eta}_{\mu'}\, dt\right)\widehat{\rho}_t - \frac{1}{2}\sum_{\mu=1}^{m}\{C_\mu^\dagger C_\mu, \widehat{\rho}_t\}\, dt - i[H, \widehat{\rho}_t]\, dt + \sum_{\mu=1}^{m}\left(1 - \sum_{\mu'=1}^{m'} \eta_{\mu',\mu}\right) C_\mu \widehat{\rho}_t C_\mu^\dagger\, dt}{1 - \sum_{\mu'=1}^{m'} \bar{\eta}_{\mu'}\, dt - \sum_{\mu=1}^{m}\sum_{\mu'=1}^{m'} \eta_{\mu',\mu} \operatorname{Tr}\left(C_\mu^\dagger \widehat{\rho}_t C_\mu\right)\, dt} \\
&= \left(\widehat{\rho}_t - \sum_{\mu'=1}^{m'} \bar{\eta}_{\mu'}\, \widehat{\rho}_t\, dt - \frac{1}{2}\sum_{\mu=1}^{m}\{C_\mu^\dagger C_\mu, \widehat{\rho}_t\}\, dt - i[H, \widehat{\rho}_t]\, dt + \sum_{\mu=1}^{m}(1 - \sum_{\mu'=1}^{m'} \eta_{\mu',\mu})\, C_\mu \widehat{\rho}_t C_\mu^\dagger\, dt\right) \times \\
&\left(1 + \sum_{\mu'=1}^{m'} \bar{\eta}_{\mu'}\, dt + \sum_{\mu=1}^{m}\sum_{\mu'=1}^{m'} \eta_{\mu',\mu} \operatorname{Tr}\left(C_\mu^\dagger \widehat{\rho}_t C_\mu\right)\, dt\right).
\end{aligned} \tag{6.5}$$

In the first order terms of dt, Equation (6.5) is equal to:

$$\begin{aligned}
\mathbb{L}_0(\widehat{\rho}_t) = \widehat{\rho}_t &- \frac{1}{2}\sum_{\mu=1}^{m}\{C_\mu^\dagger C_\mu, \widehat{\rho}_t\}\, dt - i[H, \widehat{\rho}_t]\, dt \\
&+ \sum_{\mu=1}^{m}\left(1 - \sum_{\mu'=1}^{m'} \eta_{\mu',\mu}\right) C_\mu \widehat{\rho}_t C_\mu^\dagger\, dt + \sum_{\mu=1}^{m}\sum_{\mu'=1}^{m'} \eta_{\mu',\mu} \operatorname{Tr}\left(C_\mu \widehat{\rho}_t C_\mu^\dagger\right) \widehat{\rho}_t\, dt.
\end{aligned}$$

Similarly, we can also express the superoperator $\mathbb{L}_{\mu'}$, in the first order of dt, for $\mu' \neq 0$:

$$\begin{aligned}
\mathbb{L}_{\mu'}(\widehat{\rho}_t) &= \frac{\bar{\eta}_{\mu'}\, dt\left(\widehat{\rho}_t - \frac{1}{2}\sum_{\mu=1}^{m}\{C_\mu^\dagger C_\mu, \widehat{\rho}_t\}\, dt - i[H, \widehat{\rho}_t]\, dt\right) + \sum_{\mu=1}^{m} \eta_{\mu',\mu}\, C_\mu \widehat{\rho}_t C_\mu^\dagger\, dt}{\operatorname{Tr}\left(\bar{\eta}_{\mu'}\, dt\left(\widehat{\rho}_t - \frac{1}{2}\sum_{\mu=1}^{m}\{C_\mu^\dagger C_\mu, \widehat{\rho}_t\}\, dt - i[H, \widehat{\rho}_t]\, dt\right) + \sum_{\mu=1}^{m} \eta_{\mu',\mu}\, C_\mu \widehat{\rho}_t C_\mu^\dagger\, dt\right)} \\
&= \frac{\bar{\eta}_{\mu'}\, \widehat{\rho}_t + \sum_{\mu=1}^{m} \eta_{\mu',\mu}\, C_\mu \widehat{\rho}_t C_\mu^\dagger}{\bar{\eta}_{\mu'} + \sum_{\mu=1}^{m} \eta_{\mu',\mu} \operatorname{Tr}\left(C_\mu \widehat{\rho}_t C_\mu^\dagger\right)}.
\end{aligned}$$

Hence, the stochastic evolution of the optimal filter $\widehat{\rho}_t$ can be modeled through the following Monte Carlo trajectories.

$$\widehat{\rho}_{t+dt} = \begin{cases} \mathbb{L}_0(\widehat{\rho}_t) & \text{with probability} \quad 1 - \sum_{\mu'=1}^{m'} \bar{\eta}_{\mu'}\, dt - \sum_{\mu=1}^{m} \sum_{\mu'=1}^{m'} \eta_{\mu',\mu} \operatorname{Tr}\left(C_\mu^\dagger \widehat{\rho}_t C_\mu\right) dt, \\ \mathbb{L}_{\mu'}(\widehat{\rho}_t) & \text{with probability} \quad \bar{\eta}_{\mu'}\, dt + \sum_{\mu=1}^{m} \eta_{\mu',\mu} \operatorname{Tr}\left(C_\mu \widehat{\rho}_t C_\mu^\dagger\right) dt, \quad \text{for any} \quad \mu' \neq 0. \end{cases}$$

In order to combine these $m'+1$ possibilities, we introduce the m' Poisson processes $\widehat{N}^{\mu'}(t)$ and the processes $d\widehat{N}_t^{\mu'} := \widehat{N}^{\mu'}(t+dt) - \widehat{N}^{\mu'}(t)$ which takes unity with probability $\bar{\eta}_{\mu'}\, dt + \sum_{\mu=1}^{m} \eta_{\mu',\mu} \operatorname{Tr}\left(C_\mu \widehat{\rho}_t C_\mu^\dagger\right) dt$ and zero with the complementary probability. Thus the evolution of the optimal filter is described as follows:

$$\begin{aligned} d\widehat{\rho}_t &= -i\,[H, \widehat{\rho}_t]\, dt - \frac{1}{2}\sum_{\mu=1}^{m} \{C_\mu^\dagger C_\mu, \widehat{\rho}_t\}\, dt + \sum_{\mu=1}^{m} \left(1 - \sum_{\mu'=1}^{m'} \eta_{\mu',\mu}\right) C_\mu \widehat{\rho}_t C_\mu^\dagger\, dt \\ &\quad + \sum_{\mu=1}^{m} \sum_{\mu'=1}^{m'} \eta_{\mu',\mu} \operatorname{Tr}\left(C_\mu \widehat{\rho}_t C_\mu^\dagger\right) \widehat{\rho}_t\, dt + \sum_{\mu'=1}^{m'} \widehat{\Upsilon}_{\mu'}(\widehat{\rho}_t)\, d\widehat{N}_t^{\mu'}, \end{aligned} \tag{6.6}$$

where the superoperator $\widehat{\Upsilon}_{\mu'}$ is defined as follows

$$\widehat{\Upsilon}_{\mu'}(\rho) := \frac{\bar{\eta}_{\mu'}\rho + \sum_{\mu=1}^{m} \eta_{\mu',\mu} C_\mu \rho C_\mu^\dagger}{\bar{\eta}_{\mu'} + \sum_{\mu=1}^{m} \eta_{\mu',\mu} \operatorname{Tr}\left(C_\mu \rho C_\mu^\dagger\right)} - \rho.$$

Stochastic master Equation (6.6) can also be rewritten as follows

$$\begin{aligned} d\widehat{\rho}_t &= \mathcal{L}(\widehat{\rho}_t)\, dt \\ &+ \sum_{\mu'=1}^{m'} \left(\frac{\bar{\eta}_{\mu'}\widehat{\rho}_t + \sum_{\mu=1}^{m} \eta_{\mu',\mu} C_\mu \widehat{\rho}_t C_\mu^\dagger}{\bar{\eta}_{\mu'} + \sum_{\mu=1}^{m} \eta_{\mu',\mu} \operatorname{Tr}\left(C_\mu \widehat{\rho}_t C_\mu^\dagger\right)} - \widehat{\rho}_t \right) \left(d\widehat{N}_t^{\mu'} - \bar{\eta}_{\mu'}\, dt - \sum_{\mu=1}^{m} \eta_{\mu',\mu} \operatorname{Tr}\left(C_\mu \widehat{\rho}_t C_\mu^\dagger\right) dt \right), \end{aligned} \tag{6.7}$$

with the Lindblad operator

$$\mathcal{L}(\widehat{\rho}_t) := -i\,[H, \widehat{\rho}_t] - \frac{1}{2}\sum_{\mu=1}^{m} \{C_\mu^\dagger C_\mu, \widehat{\rho}_t\} + \sum_{\mu=1}^{m} C_\mu \widehat{\rho}_t C_\mu^\dagger.$$

Notice that the average dynamics of $\widehat{\rho}$ is still governed by the Lindblad term:

$$d\langle \widehat{\rho}_t \rangle = \mathcal{L}(\langle \widehat{\rho}_t \rangle)\, dt,$$

where we have used the fact that $\langle d\widehat{N}_t^{\mu'} \rangle = \bar{\eta}_{\mu'}\, dt + \sum_{\mu=1}^{m} \eta_{\mu',\mu} \operatorname{Tr}\left(C_\mu \widehat{\rho}_t C_\mu^\dagger\right) dt$.

6.2 SMEs driven by multidimensional diffusive processes

Another type of continuous-time stochastic master equations correspond to those driven by multidimensional Wiener processes. Such diffusive type measurement back-action is particularly relevant when one considers a Homodyne detection of the scattered coherent light through the quantum system to be measured [9, 109].

6.2.1 Perfect measurement

The SME described by the p-dimensional diffusive terms is as follows (see [37, 16, 20, 103]):

$$d\rho_t = \mathcal{L}(\rho_t)\,dt + \sum_{\nu=1}^{p} \Lambda_\nu(\rho_t)\,dW_t^\nu, \tag{6.8}$$

where

- the Lindblad operator is given by

$$\mathcal{L}(\rho) := -i\,[H,\rho] + \sum_{\nu=1}^{p} \mathcal{L}_\nu(\rho),$$

  with $\mathcal{L}_\nu(\rho) := -\frac{1}{2}\{L_\nu^\dagger L_\nu, \rho\} + L_\nu \rho L_\nu^\dagger$, and L_ν is an arbitrary operator in $\mathbb{C}^{d\times d}$;

- the superoperator Λ_ν is defined as follows

$$\Lambda_\nu(\rho) := L_\nu\rho + \rho L_\nu^\dagger - \mathrm{Tr}\left((L_\nu + L_\nu^\dagger)\rho\right)\rho,$$

  for any $\nu \in \{1,\dots,p\}$;

- dW_t^ν is the Wiener process which is the following innovation

$$dW_t^\nu = dy_t^\nu - \mathrm{Tr}\left(\left(L_\nu + {L_\nu}^\dagger\right)\rho_t\right)\,dt\ ,$$

  where y_t^ν is a continuous semi-martingale corresponding to the outcome ν.

Notice that the average dynamics reads $d\langle\rho_t\rangle = \mathcal{L}(\langle\rho_t\rangle)\,dt$, since $\langle dW_t^\nu\rangle = 0$ for any $\nu \in \{1,\dots,p\}$.

An example of the diffusion process: a two level atom. We consider a single qubit in dispersive interaction with an optical probe in the z-direction and controlled by a magnetic field in the y-direction. Assuming a Homodyne detection of the probe field, one achieves, in the Markovian approximation, the following stochastic master equation [112]:

$$\begin{aligned} d\rho_t = &-iu_t\,[\sigma_y,\rho_t]\,dt + \left(\sigma_z\rho_t\sigma_z^\dagger - \frac{1}{2}\sigma_z^\dagger\sigma_z\rho_t - \frac{1}{2}\rho_t\sigma_z^\dagger\sigma_z\right)dt \\ &+ \left(\sigma_z\rho_t + \rho_t\sigma_z^\dagger - 2\mathrm{Tr}\left(\sigma_z\rho_t\right)\rho_t\right)dW_t, \end{aligned}$$

where

- $u_t \in \mathbb{R}$ is the control input at time $t > 0$;
- the controlled evolution given in above is a particular form of Equation (6.8), where $p = 1$,
$$H = u_t \sigma_y = u_t \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \text{and} \quad L_1 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$$
- the innovation term dW_t is given by $dW_t = dy_t - 2\mathrm{Tr}\,(\sigma_z \rho_t)$, where dy_t denotes the measurement record.

We remark that if we set the control to zero, the system state converges almost surely towards the state $|g\rangle\langle g|$ or $|e\rangle\langle e|$ (it is sufficient to take the following supermartingale: $-\mathrm{Tr}^2\,(\sigma_z \rho)$ as Lyapunov function). But an appropriate feedback can be applied to stabilize an arbitrary pure state of the atom either $|g\rangle\langle g|$ or $|e\rangle\langle e|$ (see [71] for more details).

6.2.2 Imperfect measurement

We take the same model of imperfection discussed in Section 6.1.2. The approach that we apply to obtain the stochastic master equation driven by multidimensional Wiener processes is based on the diffusion approximation, as in Section 5.2. Thanks to Equation (6.7), we know that the evolution of the optimal filter described by the Poisson processes takes the following form

$$d\widehat{\rho}_t = \mathcal{L}(\widehat{\rho}_t)\,dt + \sum_{\mu'=1}^{m'} \widehat{\Upsilon}_{\mu'}(\widehat{\rho}_t) \left(d\widehat{N}_t^{\mu'} - \bar{\eta}_{\mu'}\,dt - \sum_{\mu=1}^{m} \eta_{\mu',\mu} \mathrm{Tr}\left(C_\mu \widehat{\rho}_t C_\mu^\dagger\right) dt \right).$$

In fact, the total transformation that we should make to go from direct to Homodyne detection is the following transformation

$$\forall \mu \in \{1,\ldots,m\}: \quad C_\mu \longrightarrow C_\mu + \varsigma, \quad H \longrightarrow H - \tfrac{i}{2}\varsigma \sum_{\mu=1}^{m} (C_\mu - C_\mu^\dagger),$$

where $\varsigma > 0$ is a large real number (which corresponds to the amplitude of the local oscillator in Homodyne detection). Thus the above SME becomes:

$$\begin{aligned} d\widehat{\rho}_t =\; & -i[H,\widehat{\rho}_t]\,dt - \frac{1}{2}\sum_{\mu=1}^{m} [\varsigma C_\mu - \varsigma C_\mu^\dagger, \widehat{\rho}_t]\,dt \\ & + \sum_{\mu=1}^{m} \Big(-\frac{1}{2}\{(C_\mu^\dagger + \varsigma)(C_\mu + \varsigma), \widehat{\rho}_t\} + (C_\mu + \varsigma)\widehat{\rho}_t (C_\mu^\dagger + \varsigma) \Big)\,dt \\ & + \sum_{\mu'=1}^{m'} \widehat{\Upsilon}_{\mu'}^{\varsigma}(\widehat{\rho}_t) \Big(d\widehat{N}_t^{\mu',\varsigma} - \bar{\eta}_{\mu'}\,dt - \sum_{\mu=1}^{m} \eta_{\mu',\mu} \mathrm{Tr}\left((C_\mu + \varsigma)\widehat{\rho}_t (C_\mu^\dagger + \varsigma)\right) dt \Big), \end{aligned} \tag{6.9}$$

where the superoperator $\widehat{\Upsilon}^{\varsigma}_{\mu'}$ is defined as follows

$$\widehat{\Upsilon}^{\varsigma}_{\mu'}(\rho) := \frac{\bar{\eta}_{\mu'}\rho + \sum_{\mu=1}^{m}\eta_{\mu',\mu}(C_\mu+\varsigma)\rho(C_\mu^\dagger+\varsigma)}{\bar{\eta}_{\mu'} + \sum_{\mu=1}^{m}\eta_{\mu',\mu}\mathrm{Tr}\Big((C_\mu+\varsigma)\rho(C_\mu^\dagger+\varsigma)\Big)} - \rho.$$

The process $d\widehat{N}^{\mu',\varsigma}_t := \widehat{N}^{\mu',\varsigma}(t+dt) - \widehat{N}^{\mu',\varsigma}(t)$ (where $\widehat{N}^{\mu',\varsigma}(t)$ are the m' Poisson processes) takes unity with probability $\bar{\eta}_{\mu'}\,dt + \sum_{\mu=1}^{m}\eta_{\mu',\mu}\mathrm{Tr}\left((C_\mu+\varsigma)\widehat{\rho}_t(C_\mu^\dagger+\varsigma)\right)\,dt$ and zero with the complementary probability. Now we take the limit $\varsigma \longrightarrow \infty$ (corresponding to a strong local oscillator). By the central limit theorem, we have

$$d\widehat{N}^{\mu',\varsigma}_t \xrightarrow{\text{law}} \langle \frac{d\widehat{N}^{\mu',\varsigma}_t}{dt}\rangle\,dt + \sqrt{\langle \frac{d\widehat{N}^{\mu',\varsigma}_t}{dt}\rangle}\,d\widehat{W}^{\mu'}_t,$$

thus we find the following limit, for any $\mu' \in \{1,\dots,m'\}$:

$$\begin{aligned} d\widehat{N}^{\mu',\varsigma}_t \xrightarrow{\text{law}}\ & \bar{\eta}_{\mu'}\,dt + \sum_{\mu=1}^{m}\eta_{\mu',\mu}\mathrm{Tr}\Big((C_\mu+\varsigma)\widehat{\rho}_t(C_\mu^\dagger+\varsigma)\Big)\,dt \\ & + \sqrt{\bar{\eta}_{\mu'} + \sum_{\mu=1}^{m}\eta_{\mu',\mu}\mathrm{Tr}\Big((C_\mu+\varsigma)\widehat{\rho}_t(C_\mu^\dagger+\varsigma)\Big)}\,d\widehat{W}^{\mu'}_t. \end{aligned}$$

Replacing $d\widehat{N}^{\mu',\varsigma}_t$ appeared in Equation (6.9) by its limit given above and simplifying the terms in ς and ς^2, we find

$$d\widehat{\rho}_t = \mathcal{L}(\widehat{\rho}_t)\,dt + \sum_{\mu'=1}^{m'}\widehat{\Upsilon}^{\varsigma}_{\mu'}(\widehat{\rho}_t)\sqrt{\bar{\eta}_{\mu'} + \sum_{\mu=1}^{m}\eta_{\mu',\mu}\mathrm{Tr}\Big(C_\mu+\varsigma)\widehat{\rho}_t(C_\mu^\dagger+\varsigma)\Big)}\,d\widehat{W}^{\mu'}_t. \tag{6.10}$$

Expanding the diffusive term and keeping only the terms in order of ς^{-1}, we find

$$\begin{aligned} &\widehat{\Upsilon}^{\varsigma}_{\mu'}(\widehat{\rho}_t)\sqrt{\bar{\eta}_{\mu'} + \sum_{\mu=1}^{m}\eta_{\mu',\mu}\mathrm{Tr}\Big((C_\mu+\varsigma)\widehat{\rho}_t(C_\mu^\dagger+\varsigma)\Big)}\,d\widehat{W}^{\mu'}_t \\ &= \left(\frac{\bar{\eta}_{\mu'}\widehat{\rho}_t + \sum_{\mu=1}^{m}\eta_{\mu',\mu}(C_\mu+\varsigma)\widehat{\rho}_t(C_\mu^\dagger+\varsigma)}{\bar{\eta}_{\mu'} + \sum_{\mu=1}^{m}\eta_{\mu',\mu}\mathrm{Tr}\Big((C_\mu+\varsigma)\widehat{\rho}_t(C_\mu^\dagger+\varsigma)\Big)} - \widehat{\rho}_t\right)\sqrt{\bar{\eta}_{\mu'} + \sum_{\mu=1}^{m}\eta_{\mu',\mu}\mathrm{Tr}\Big((C_\mu+\varsigma)\widehat{\rho}_t(C_\mu^\dagger+\varsigma)\Big)}\,d\widehat{W}^{\mu'}_t \\ &= \left(\frac{\varsigma^2\sum_{\mu=1}^{m}\eta_{\mu',\mu}\widehat{\rho}_t + \varsigma\sum_{\mu=1}^{m}\eta_{\mu',\mu}\big(C_\mu\widehat{\rho}_t + \widehat{\rho}_t C_\mu^\dagger\big) - \sum_{\mu=1}^{m}\eta_{\mu',\mu}\mathrm{Tr}\Big((\varsigma C_\mu + \varsigma C_\mu^\dagger)\widehat{\rho}_t\Big)\widehat{\rho}_t}{\sum_{\mu=1}^{m}\eta_{\mu',\mu}\varsigma^2} - \widehat{\rho}_t\right) \times \\ &\sqrt{\sum_{\mu=1}^{m}\eta_{\mu',\mu}\varsigma^2}\,d\widehat{W}^{\mu'}_t = \frac{\sum_{\mu=1}^{m}\eta_{\mu',\mu}\Big(C_\mu\widehat{\rho}_t + \widehat{\rho}_t C_\mu^\dagger - \mathrm{Tr}\Big((C_\mu + C_\mu^\dagger)\widehat{\rho}_t\Big)\widehat{\rho}_t\Big)}{\sum_{\mu=1}^{m}\eta_{\mu',\mu}}\sqrt{\sum_{\mu=1}^{m}\eta_{\mu',\mu}}\,d\widehat{W}^{\mu'}_t =: \\ &\widehat{\Lambda}_{\mu'}(\widehat{\rho}_t)\,d\widehat{W}^{\mu'}_t, \end{aligned}$$

where $\widehat{\Lambda}_{\mu'}$ is defined as follows

$$\widehat{\Lambda}_{\mu'}(\rho) := \widehat{C}_{\mu'}\rho + \rho\widehat{C}_{\mu'}^{\dagger} - \mathrm{Tr}\left((\widehat{C}_{\mu'} + \widehat{C}_{\mu'}^{\dagger})\rho\right)\rho$$

with $\widehat{C}_{\mu'} = \frac{\sum_{\mu=1}^{m} \eta_{\mu',\mu} C_\mu}{\sum_{\mu=1}^{m} \eta_{\mu',\mu}}$. Thus SME (6.10) becomes

$$d\widehat{\rho}_t = \mathcal{L}(\widehat{\rho}_t)\, dt + \sum_{\mu'=1}^{m'} \widehat{\Lambda}_{\mu'}(\widehat{\rho}_t)\, d\widehat{W}_t^{\mu'},$$

where $d\widehat{W}_t^{\mu'}$ is the Wiener process given by

$$d\widehat{W}_t^{\mu'} = d\widehat{y}_t^{\mu'} - \sqrt{\sum_{\mu} \eta_{\mu',\mu}} \mathrm{Tr}\left(\widehat{C}_{\mu'}\widehat{\rho}_t\widehat{C}_{\mu'}^{\dagger}\right)\, dt, \tag{6.11}$$

and $\widehat{y}_t^{\mu'}$ is the measurement record corresponding to the outcome μ'.

6.3 Conclusion

In this section, we consider a generic stochastic master equation driven simultaneously by m_P-dimensional Poisson and m_W-dimensional Wiener processes.

Perfect measurement

The evolution of an open quantum system can be described by the following SME (see [10, 72]):

$$d\rho_t = \mathcal{L}(\rho_t)\, dt + \sum_{\nu=1}^{m_W} \Lambda_\nu(\rho_t)\, dW_t^\nu + \sum_{\mu=1}^{m_P} \Upsilon_\mu(\rho_t)\, \left(dN_t^\mu - \mathrm{Tr}\left(C_\mu\rho_t C_\mu^\dagger\right)\, dt\right), \tag{6.12}$$

where

- the Lindblad operator is given by the following expression

$$\mathcal{L}(\rho_t) := -i\,[H, \rho_t] + \sum_{\mu=1}^{m_P} \mathcal{L}_\mu^P(\rho_t) + \sum_{\nu=1}^{m_W} \mathcal{L}_\nu^W(\rho_t), \tag{6.13}$$

  with the superoperators $\mathcal{L}_\mu^P$ and $\mathcal{L}_\nu^W$ respectively defined by

$$\mathcal{L}_\mu^P(\rho) := -\frac{1}{2}\{C_\mu^\dagger C_\mu, \rho\} + C_\mu\rho C_\mu^\dagger \quad \text{and} \quad \mathcal{L}_\nu^W(\rho) := -\frac{1}{2}\{L_\nu^\dagger L_\nu, \rho\} + L_\nu\rho L_\nu^\dagger;$$

- the superoperators Υ_μ and Λ_ν have the following forms

$$\Upsilon_\mu(\rho) := \frac{C_\mu \rho C_\mu^\dagger}{\mathrm{Tr}\left(C_\mu \rho C_\mu^\dagger\right)} - \rho \quad \text{and} \quad \Lambda_\nu(\rho) := L_\nu \rho + \rho L_\nu^\dagger - \mathrm{Tr}\left((L_\nu + L_\nu^\dagger)\rho\right)\rho;$$

- the jump detector μ corresponds to the Poisson process $N^\mu(t)$, where $dN_t^\mu = N^\mu(t + dt) - N^\mu(t) = 1$ happens with probability $\mathrm{Tr}\left(C_\mu \rho_t {C_\mu}^\dagger\right) dt$;

- the continuous detector ν refers to the continuous signal y_t^ν related to the Wiener process dW_t^ν by

$$dy_t^\nu = dW_t^\nu + \mathrm{Tr}\left((L_\nu + L_\nu^\dagger)\rho_t\right) dt. \tag{6.14}$$

The evolution of the quantum filter of state $\rho_t^e \in \mathcal{D}$ is given by the following stochastic master equation which depends on the true quantum state ρ_t via relation (6.14)

$$\begin{aligned} d\rho_t^e = \mathcal{L}(\rho_t^e)\, dt &+ \sum_{\nu=1}^{m_W} \Lambda_\nu(\rho_t^e) \left(dy_t^\nu - \mathrm{Tr}\left((L_\nu + L_\nu^\dagger)\rho_t^e\right) dt\right) \\ &+ \sum_{\mu=1}^{m_P} \Upsilon_\mu(\rho_t^e) \left(dN_t^\mu - \mathrm{Tr}\left(C_\mu \rho_t^e C_\mu^\dagger\right) dt\right). \end{aligned} \tag{6.15}$$

So, this can be rewritten as follows

$$\begin{aligned} d\rho_t^e = \mathcal{L}(\rho_t^e)\, dt &+ \sum_{\nu=1}^{m_W} \Lambda_\nu(\rho_t^e) \left(\mathrm{Tr}\left((L_\nu + L_\nu^\dagger)\rho_t\right) - \mathrm{Tr}\left((L_\nu + L_\nu^\dagger)\rho_t^e\right)\right) dt \\ &+ \sum_{\nu=1}^{m_W} \Lambda_\nu(\rho_t^e) dW_t^\nu + \sum_{\mu=1}^{m_P} \Upsilon_\mu(\rho_t^e) \left(dN_t^\mu - \mathrm{Tr}\left(C_\mu \rho_t^e C_\mu^\dagger\right) dt\right). \end{aligned}$$

Note that if ρ_0 and ρ_0^e are in $\mathcal{D}$, then ρ_t and ρ_t^e remain in $\mathcal{D}$ for all $t > 0$. If the initial states are pure, i.e., $\mathrm{Tr}\,(\rho_0^2) = \mathrm{Tr}\,((\rho_0^e)^2) = 1$, this purity will not be changed during the time evolutions, i.e., $\mathrm{Tr}\,(\rho_t^2) = \mathrm{Tr}\,((\rho_t^e)^2) = 1$ for all $t > 0$.

Imperfect measurement

Let us now define the new notations in the aim of modeling the imperfect jump-diffusion SMEs. Set the real outcomes $\mu' \in \{0, 1, \dots, m'_P\}$ and the ideal outcomes $\mu \in \{0, 1, \dots, m_P\}$ for the measurements associated to the jump terms. The classical stochastic model relating to the real and ideal outcomes is parameterized by the $(m'_P + 1) \times m_P$ left stochastic matrix $\eta^P = (\eta^P_{\mu',\mu})_{0 \le \mu' \le m'_P, 1 \le \mu \le m_P}$ and the positive vector $\bar{\eta}^P = (\bar{\eta}^P_{\mu'})_{1 \le \mu' \le m'_P}$ in $\mathbb{R}_+^{m'_P}$.

Similarly we take the following notations associated to the diffusive terms. The m'_W real continuous signals $y_t^{\nu'}$ with $\nu' \in \{1, \dots, m'_W\}$, and the m_W ideal ones y_t^ν with $\nu \in$

$\{1,\dots,m_W\}$ are correlated by the $m'_W \times m_W$ matrix $\eta^W = (\eta^W_{\nu',\nu})_{1\leq \nu'\leq m'_W, 1\leq \nu \leq m_W}$, with $0 \leq \eta_{\nu',\nu} \leq 1$ and $\sum_{\nu'=1}^{m'_W} \eta_{\nu',\nu} \leq 1$.

With these new notations, we get the following jump-diffusion SME for the optimal filter:

$$d\widehat{\rho}_t = \mathcal{L}(\widehat{\rho}_t)\,dt + \sum_{\mu'=1}^{m'_P} \widehat{\Upsilon}_{\mu'}(\widehat{\rho}_t)\left(d\widehat{N}^{\mu'}_t - \bar{\eta}^P_{\mu'}\,dt - \sum_{\mu=1}^{m_P} \eta^P_{\mu',\mu}\mathrm{Tr}\left(C_\mu \widehat{\rho}_t C^\dagger_\mu\right)\,dt\right) + \sum_{\nu'=1}^{m'_W} \sqrt{\bar{\eta}^W_{\nu'}}\widehat{\Lambda}_{\nu'}(\widehat{\rho}_t)\,d\widehat{W}^{\nu'}_t, \quad (6.16)$$

with $\bar{\eta}^W_{\nu'} = \sum_{\nu=1}^{m_W} \eta^W_{\nu',\nu}$. Now we recall the notations applied in the above SME:

- the Lindblad operator $\mathcal{L}$ is defined by (6.13)
- the superoperators $\widehat{\Upsilon}_{\mu'}$ and $\widehat{\Lambda}_{\nu'}$ are given by

  $$\widehat{\Upsilon}_{\mu'}(\rho) := \frac{\bar{\eta}^P_{\mu'}\rho + \sum_{\mu=1}^{m_P}\eta^P_{\mu',\mu}C_\mu\rho C^\dagger_\mu}{\bar{\eta}^P_{\mu'} + \sum_{\mu=1}^{m_P}\eta^P_{\mu',\mu}\mathrm{Tr}\left(C_\mu\rho C^\dagger_\mu\right)} - \rho \quad \text{and} \quad \widehat{\Lambda}_{\nu'}(\rho) = \widehat{L}_{\nu'}\rho + \rho\widehat{L}^\dagger_{\nu'} - \mathrm{Tr}\left((\widehat{L}_{\nu'} + \widehat{L}^\dagger_{\nu'})\rho\right)\rho,$$

  with $\widehat{L}_{\nu'} := (\sum_{\nu=1}^{m_W}\eta^W_{\nu',\nu}L_\nu)/\bar{\eta}^W_{\nu'}$;
- the jump detector μ' corresponds to the Poisson process $\widehat{N}^{\mu'}(t)$, where $d\widehat{N}^{\mu'}_t = \widehat{N}^{\mu'}(t+dt) - \widehat{N}^{\mu'}(t) = 1$ happens with probability

  $$\widehat{P}_{\mu'}(\widehat{\rho}_t) = \bar{\eta}^P_{\mu'}\,dt + \sum_{\mu=1}^{m_P}\eta^P_{\mu',\mu}\mathrm{Tr}\left(C_\mu\widehat{\rho}_t C^\dagger_\mu\right)\,dt; \quad (6.17)$$

- the continuous detector ν' refers to the continuous signal $\widehat{y}^{\nu'}_t$ related to the Wiener process $d\widehat{W}^{\nu'}_t$ by

  $$d\widehat{y}^{\nu'}_t = d\widehat{W}^{\nu'}_t + \sqrt{\bar{\eta}^W_{\nu'}}\mathrm{Tr}\left((\widehat{L}_{\nu'} + \widehat{L}^\dagger_{\nu'})\widehat{\rho}_t\right)\,dt. \quad (6.18)$$

The estimate optimal filter of state $\widehat{\rho}^e_t$ is governed by the following stochastic master equation which depends on the optimal filter state $\widehat{\rho}_t$ via (6.17) and (6.18).

$$d\widehat{\rho}^e_t = \mathcal{L}(\widehat{\rho}^e_t)\,dt + \sum_{\mu'=1}^{m'_P}\widehat{\Upsilon}_{\mu'}(\widehat{\rho}^e_t)\left(d\widehat{N}^{\mu'}_t - \bar{\eta}^P_{\mu'}\,dt - \sum_{\mu=1}^{m_P}\eta^P_{\mu',\mu}\mathrm{Tr}\left(C_\mu\widehat{\rho}^e_t C^\dagger_\mu\right)\,dt\right) + \sum_{\nu'=1}^{m'_W}\sqrt{\bar{\eta}^W_{\nu'}}\widehat{\Lambda}_{\nu'}(\widehat{\rho}^e_t)\left(d\widehat{y}^{\nu'}_t - \sqrt{\bar{\eta}^W_{\nu'}}\mathrm{Tr}\left((\widehat{L}_{\nu'} + \widehat{L}^\dagger_{\nu'})\widehat{\rho}^e_t\right)\,dt\right). \quad (6.19)$$

Replacing $d\widehat{y}_t^{\nu'}$ by its value given in (6.18) in the above equation, we find

$$d\widehat{\rho}_t^e = \mathcal{L}(\widehat{\rho}_t^e)\,dt + \sum_{\mu'=1}^{m'_P} \widehat{\Upsilon}_{\mu'}(\widehat{\rho}_t^e)\left(d\widehat{N}_t^{\mu'} - \bar{\eta}_{\mu'}^P\,dt - \sum_{\mu=1}^{m_P} \eta_{\mu',\mu}^P \mathrm{Tr}\left(C_\mu \widehat{\rho}_t^e C_\mu^\dagger\right)\,dt\right)$$
$$+ \sum_{\nu'=1}^{m'_W} \bar{\eta}_{\nu'}^W \widehat{\Lambda}_{\nu'}(\widehat{\rho}_t^e)\left(\mathrm{Tr}\left((\widehat{L}_{\nu'} + \widehat{L}_{\nu'}^\dagger)\widehat{\rho}_t\right) - \mathrm{Tr}\left((\widehat{L}_{\nu'} + \widehat{L}_{\nu'}^\dagger)\widehat{\rho}_t^e\right)\right)\,dt + \sum_{\nu'=1}^{m'_W} \sqrt{\bar{\eta}_{\nu'}^W}\,\widehat{\Lambda}_{\nu'}(\widehat{\rho}_t^e)d\widehat{W}_t^{\nu'}.$$

We remark that as soon as ρ_0 and ρ_0^e belong to $\mathcal{D}$, ρ_t and ρ_t^e remain in $\mathcal{D}$ for all $t \geq 0$. Contrarily to the case of perfect measurement, if the initial states $\widehat{\rho}_0$ and $\widehat{\rho}_0^e$ are pure, this will not be necessarily preserved through the evolutions (6.16) and (6.19).

We can now state the following conjecture whose proof is in progress [4].

Conjecture 1. *Consider the Markov processes $(\widehat{\rho}_t, \widehat{\rho}_t^e)$ satisfying stochastic master Equations (6.16) and (6.19) respectively with $\widehat{\rho}_0$, $\widehat{\rho}_0^e$ in $\mathcal{D}$. Then the fidelity $F(\widehat{\rho}, \widehat{\rho}^e)$, defined by (5.4) is a submartingale, i.e., $\mathbb{E}\left[F(\widehat{\rho}_t, \widehat{\rho}_t^e)|(\widehat{\rho}_s, \widehat{\rho}_s^e)\right] \geq F(\widehat{\rho}_s, \widehat{\rho}_s^e)$, for all $t \geq s$.*

Sketch of a possible proof. We can proceed in the same way as for the proof of Theorem 5.1. In the first step, we describe how we obtain stochastic master Equations (6.16) and (6.19) as the limits of the SMEs driven only by the Poisson processes (diffusion approximation). In second step, we show the analogue of Proposition 5.1 which ensures that the fidelity between two arbitrary solutions of the SMEs with Poisson processes is a submartingale. This can be proved thanks to Theorem 2.3 showing the stability of the estimate optimal filter for discrete-time imperfect measurements which are POVM. Finally, an analogue of Theorem 5.2 (convergence in law) could be stated to conclude that the fidelity between $\widehat{\rho}$ and $\widehat{\rho}^e$ is a submartingale.

Conclusion

Cette thèse traite de la stabilisation de systèmes quantiques par des feedbacks basés sur des mesures QND. Elle porte également sur la stabilité de filtres quantiques en temps continu et l'élaboration de SMEs saut-diffusion en présence d'erreurs de mesure.

La Partie I de cette thèse était dédiée aux systèmes quantiques ouverts à temps discret. Dans le Chapitre 3, nous avons proposé un schéma de rétroaction basée sur des mesures visant à stabiliser un système physique, consistant en une cavité micro-onde sondée par des atomes de Rydberg, vers un certain état de Fock. Ce type de schéma de rétroaction qui tient compte d'un délai pur est basé sur une approche de Lyapunov. Dans le Chapitre 4, nous avons considéré plus généralement les systèmes quantiques en temps discret sujet à des mesures QND. L'objectif de ce chapitre était de concevoir une rétroaction qui puisse stabiliser de manière déterministe le système considéré vers un état pur prédéfini. De plus, nous avons proposé des fonctions de Lyapunov qui rendent l'analyse de la convergence plus facile car l'utilisation du principe d'invariance de Kushner n'est pas nécessaire. En outre, nous avons prouvé la convergence d'une rétroaction qui tient compte de certains délais et d'imperfections dans les mesures. L'efficacité d'une telle rétroaction a été testée expérimentalement pour le cas de la boite à photons au Laboratoire Kastler Brossel (LKB) de l'École Normale Supérieure (ENS) de Paris.

Les paragraphes suivants présentent des extensions possibles de la recherche élaborée dans la première partie.

- Considérons une rétroaction de stabilisation appliquée à une évolution en temps continu, comme celle étudiée en [71]. Dans ce travail, une stratégie systématique de rétroaction a été proposée pour assurer la stabilisation d'états intriqués symétriques à plusieurs qubits (i.e., des états qui sont invariants par permutation des atomes dans l'ensemble). Dans ce but, les auteurs proposent une méthode de Lyapunov. Cependant, la stabilisation obtenue n'est pas fondée sur une fonction de Lyapunov stricte et il faut donc appliquer le théorème d'invariance de Kushner [55] pour assurer la stabilité asymptotique. Nous pensons qu'il est possible de stabiliser n'importe quels états intriqués en utilisant les techniques présentées au Chapitre 4 en construisant une fonction de Lyapunov stricte. Nous notons qu'il est impossible d'utiliser une approche similaire à celle du Chapitre 4 pour le temps continu, puisque la minimisation conduit à un feedback qui pourrait ne pas être continu par rapport à l'état considéré. Ainsi, la traduction de la technique de rétroaction considérée au Chapitre 4 pour le temps continu n'est pas évidente : l'existence et l'unicité de trajectoires en boucle fermée sont problématiques.

- La méthode de stabilisation de [71] est basée sur la simulation en temps réel de l'équation d'un filtre quantique pour obtenir une estimation de l'état quantique. Cependant, cette équation de filtre quantique est en général de grande dimension (dimension $N+1$ en considérant des états symétriques de N systèmes de spin-1/2 de dimension 2^N si nous-nous intéressons à tous les états symétriques et non symétriques). Ceci, en général, rend impossible (à cause des échelles de temps trop courtes de ce système quantique) de réaliser une simulation en temps réel de l'équation du filtre. Mais la simplicité des lois de rétroactions proposées suggère qu'il puisse y avoir une équation réduite du filtre (de dimension réduite) [74] qui puisse fournir suffisamment d'information pour obtenir de telles lois de rétroaction. Une ouverture possible serait l'examen de ces problèmes de réduction de filtre en parallèle de la recherche de nouvelles lois de rétroaction pour des états non symétriques intriqués.

- Des stratégies de rétroaction quantique similaires à celles présentées dans le Chapitre 4 peuvent être appliquées pour stabiliser des états encore plus exotiques, tels que les états de radiation du chat de Schrödinger. Dans [86] et [85], les états comprimés et la superposition d'états du champ mésoscopique sont stabilisés par l'ingénierie de réservoir [79]. Il serait intéressant d'étudier une stabilisation de tels états quantiques par rétroaction basée sur des mesures.

- Supposons que nous avons des paramètres incertains, tels que des constantes d'interaction atome-champs, l'intensité de mesure et l'efficacité de détection. Pour pouvoir obtenir une méthode plus robuste, nous pouvons utiliser des techniques de contrôle adaptatif permettant d'estimer ces paramètres, tout en contrôlant le système pour qu'il converge vers l'état désiré.

Dans la Partie II, nous-nous sommes concentrés sur des systèmes quantiques ouverts à temps continu. Au cours du Chapitre 5, nous avons montré que le filtre associé à l'équation maîtresse stochastique découlant seulement par un processus de Wiener est une sous-martingale, ce qui assure la stabilité d'un tel filtre. Ceci signifie que la fidélité entre l'état physique et son filtre quantique associé est croissante en moyenne. Enfin, au Chapitre 6, nous avons obtenu par des arguments heuristiques une forme générique de SMEs mélangeant aussi bien des processus de Poisson et de Wiener multidimensionnels, en présence de quelques imperfections classiques de mesure. De plus, nous avons déterminé la stabilité des filtres quantiques associés.

Dans les paragraphes suivants, nous résumons quelques extensions possibles des recherches en rapport avec la deuxième partie.

- Notre objectif futur [4] est d'adapter l'approche utilisée dans [77] pour obtenir rigoureusement les équations maîtresses stochastiques saut-diffusion en présence de certaines imperfections classiques de mesure introduites au Chapitre 6. La preuve rigoureuse associée à la Conjecture 1 a été menée en deux étapes. La première consister en la recherche d'approximations markoviennes en temps discret et de dynamiques des filtres associés convergeant en distribution vers des équations maîtresses stochastiques saut-diffusion, en présence d'imperfection de mesures. Ensuite, à la deuxième étape, nous avons appliqué le Théorème 2.3 qui a prouvé la stabilité des filtres quantiques approximés en temps discret.

- Il serait intéressant de poursuivre la recherche en caractérisant des conditions suffisantes pour assurer la convergence asymptotique des filtres quantiques [101].

This thesis considers measurement-based feedbacks stabilizing discrete-time quantum systems which are subject to QND measurements. It is also concerned with the stability of continuous-time quantum filters and design of jump-diffusion SMEs in presence of measurement errors.

Part I of this thesis was devoted to discrete-time open quantum systems. In Chapter 3, we proposed a measurement-based feedback scheme which stabilizes a physical system, consisting of a microwave cavity probed by Rydberg atoms, towards a desired Fock state. This type of feedback which takes into account a pure delay is based on a Lyapunov approach. In Chapter 4, we considered the more general discrete-time quantum systems which are subject to QND measurements. The aim of this chapter was to design a feedback which deterministically stabilizes the system under consideration towards some predetermined pure states. Moreover, we proposed Lyapunov functions which make the convergence analysis easier since the use of Kushner's invariance principle is not necessary. In addition, we proved the convergence of a stabilizing feedback which takes into account some delays and imperfections in measurements. The efficiency of such feedback was experimentally tested for the photon box case in Laboratoire Kastler Brossel (LKB) at Ecole Normale Supérieure (ENS) de Paris.

Here we list some possible future directions related to Part I.

- Consider a stabilizing feedback applied to continuous-time evolution, as the one investigated in [71]. In this work, a systematic feedback strategy was proposed to ensure the stabilization of symmetric multi-qubit entangled states (i.e., states that are invariant under permutations of the atoms in the ensemble). In this aim, the authors propose a Lyapunov-based method. However their stabilization is not based on a strict Lyapunov function and therefore they need to apply the Kushner's invariance theorem [55] to ensure the asymptotic stability. We believe that it should be possible to stabilize any entangled states, applying the techniques given in Chapter 4 to construct a strict Lyapunov function. We note that taking a feedback similar to the one proposed in Chapter 4 in continuous-time is not possible, since the minimization produces a feedback that could be not continuous with respect to the state. So translating the feedback design considered in Chapter 4 to continuous-time is not at all straightforward: the existence and the uniqueness of closed-loop trajectories are problematic.

- The stabilization method of [71] is based on the real-time simulation of a quantum filter equation to obtain an estimate of the quantum state. However, this quantum filter equation is in general high-dimensional (dimension $N+1$ if we are interested in the symmetric states of N spin-1/2 systems and dimension 2^N if we are interested in all symmetric or non-symmetric states). This, in general, makes it impossible (due to the short time scales of the quantum system) to achieve real-time simulation of the filter equation. However, the simplicity of the proposed feedback laws suggests that there might be a reduced filter equation (of low dimension) [74] which can provide enough information to obtain these feedback laws. A possible direction could be to investigate this filter reduction problem in parallel to the research on new feedback laws for non-symmetric entangled states.

- Similar quantum feedback strategies as those considered in Chapter 4 may be applied to stabilize even more exotic states such as Schrödinger cat states of radiation. In [86] and [85] the squeezed states and the mesoscopic field state superpositions are stabilized by reservoir engineering [79]. It would be interesting to investigate a measurement-based feedback stabilization of such quantum states.

- Assume that we have some uncertain parameters such as atom-field interaction constants, measurement intensity and detection efficiency. In order to have the largest robustness, we can use adaptive control techniques allowing to estimate these parameters, while controlling the system towards the desired state.

In Part II, we focused on continuous-time open quantum systems. In Chapter 5, we showed that the filter associated to a stochastic master equation driven only by a Wiener process is a submartingale, which ensures the stability of such filter. This means that the fidelity between the physical state and its associated quantum filter increases in average. Finally, in Chapter 6, we obtained through heuristic arguments, a generic form of SMEs driven by both multidimensional Poisson and Wiener processes, in presence of some classical measurement imperfections. Moreover, we proclaimed the stability of their associated quantum filters.

Below, we resume some possible research directions related to Part II.

- Our future aim [4] is to adapt the approach used in [77] to obtain rigourously the jump-diffusion stochastic master equations in presence of some classical measurement imperfections introduced in Chapter 6. The rigorous proof associated to Conjecture 1 could be done in two following steps. The first step is to find the discrete-time Markov approximations and their associated filter dynamics which converge in distribution to jump-diffusion stochastic master equations, in presence of measurement imperfections. Then, in the second step, we apply Theorem 2.3 which proves the stability of the approximated discrete-time quantum filters.

- An important research direction is to characterize sufficient conditions ensuring the asymptotic convergence of the quantum filters [101].

Appendix A

Stability of stochastic processes

In this appendix, we give different notions of stability, some stability theorems for stochastic processes, and the definition of Lyapunov exponent of linear stochastic processes.

A.1 Different notions of stability

We start by recalling different notions of stability.

Definition A.1 (Stability in probability). *Let x_t^z be a diffusion process on the metric space S, started at $x_0 = z$, and let $\widetilde{z}$ be an equilibrium position of the diffusion, i.e., $x_t^{\widetilde{z}} = \widetilde{z}$. Then the equilibrium $\widetilde{z}$ is stable in probability if*

$$\lim_{z \longrightarrow \widetilde{z}} \mathbb{P}\left(\underset{0 \leq t < \infty}{sup} \parallel x_t^z - \widetilde{z} \parallel \geq \epsilon \right) = 0 \quad \forall \epsilon > 0.$$

Definition A.2 (Global stability). *The equilibrium $\widetilde{z}$ is said globally stable if it is stable in probability and additionally*

$$\mathbb{P}\left(\underset{t \longrightarrow \infty}{lim} x_t^z = \widetilde{z} \right) = 1, \quad \forall z \in S.$$

Definition A.3 (Lyapunov stability). *The equilibrium $\widetilde{z}$ is said Lyapunov stable if*

$$\forall \epsilon > 0: \quad \exists \delta > 0 \quad \textit{such that, if} \quad \parallel x_0^z - \widetilde{z} \parallel < \delta, \quad \textit{then} \quad \parallel x_t^z - \widetilde{z} \parallel < \epsilon.$$

Definition A.4 (Asymptotic stability). *The equilibrium $\widetilde{z}$ is said asymptotically stable if $\parallel x_0^z - \widetilde{z} \parallel < \delta$, then* $\lim_{t \longrightarrow \infty} \parallel x_t^z - \widetilde{z} \parallel = 0$.

Definition A.5 (Exponential stability). *The equilibrium $\widetilde{z}$ is said exponentially stable if it is asymptotically stable and additionally*

$$\exists \alpha, \beta, \delta > 0 \quad \textit{such that, if} \quad \parallel x_0^z - \widetilde{z} \parallel < \delta \quad \textit{then} \quad \parallel x_t^z - \widetilde{z} \parallel \leq \alpha \parallel x_0^z - \widetilde{z} \parallel \exp^{-\beta t}.$$

A.2 Stability theorems

Consider $(X_n)_{n=1}^{\infty}$ a sequence of random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and taking values in a metric space S. Let $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}$ be a nondecreasing family of sub-σ-algebras.

Definition A.6. *The sequence $(X_n, \mathcal{F}_n)_{n=1}^{\infty}$ is called respectively a supermartingale, a submartingale or a martingale, if $\mathbb{E}[|X_n|] < \infty$ for $n = 1, 2, \ldots$ and*

$$\mathbb{E}[X_n|\mathcal{F}_m] \leq X_m, \quad \textit{for} \quad n \geq m$$

$$\mathbb{E}[X_n|\mathcal{F}_m] \geq X_m, \quad \textit{for} \quad n \geq m$$

$$\mathbb{E}[X_n|\mathcal{F}_m] = X_m, \quad \textit{for} \quad n \geq m.$$

The following theorem characterizes the convergence of bounded martingales.

Theorem A.1 (Doob's first martingale convergence theorem)**.** *Let $\{X_n\}$ be a Markov chain on state space S and suppose that*

$$\mathbb{E}[X_n] \geq \mathbb{E}[X_m], \qquad \textit{for } n \geq m,$$

thus X_n is a submartingale. Denote by x^+ the positive part of x. Assume furthermore that

$$\sup_n \mathbb{E}\left[X_n^+\right] < \infty.$$

Then $\lim_n X_n$ $(= X_\infty)$ exists with probability one, and $\mathbb{E}[X_\infty^+] < \infty$.

For a proof, we refer to [59, Chapter 2].

Now we recall two results that are often referred as the stochastic versions of the Lyapunov stability theory and the LaSalle's invariance principle. For detailed discussions and proofs, we refer to [55, Sections 8.4 and 8.5].

Theorem A.2 (Doob's inequality)**.** *Let $\{X_n\}$ be a Markov chain on state space S. Suppose that there exists a non-negative function $V(x)$ satisfying*

$$\mathbb{E}[V(X_1) \mid X_0 = x] - V(x) = -k(x),$$

where $k(x) \geq 0$ on the set $\{x : V(x) < \lambda\} \equiv Q_\lambda$. Then

$$\mathbb{P}\left(\sup_{0 \leq n < \infty} V(X_n) \geq \lambda \mid X_0 = x\right) \leq \frac{V(x)}{\lambda}.$$

For the statement of the second theorem, we need to use the language of probability measures rather than of random processes. Therefore, we deal with the space $\mathcal{M}$ of probability measures on the state space S. Let $\mu_0 = \sigma$ be the initial probability distribution (everywhere through this thesis we have dealt with the case where μ_0 is a Dirac on a state ρ_0 of the state space of density matrices). Then, the probability distribution of X_n, given initial distribution σ, is to be denoted by $\mu_n(\sigma)$. Note that for $m \geq 0$, the Markov property implies: $\mu_{n+m}(\sigma) = \mu_n(\mu_m(\sigma))$.

Theorem A.3 (Kushner's invariance theorem). *Consider the same assumptions as that of Theorem A.2. Let $\mu_0 = \sigma$ be concentrated on a state $x_0 \in Q_\lambda$ (Q_λ being defined as in Theorem A.2), i.e., $\sigma(x_0) = 1$. Assume that $0 \leq k(X_n) \to 0$ in Q_λ implies that $X_n \to \{x \mid k(x) = 0\} \cap Q_\lambda \equiv K_\lambda$. Under the conditions of Theorem A.2, for trajectories never leaving Q_λ, X_n converges to K_λ almost surely. Also, the associated conditioned probability measures $\tilde{\mu}_n$ tend to the largest invariant set of measures $\mathcal{M}_\infty \subset \mathcal{M}$ whose support set is in K_λ. Finally, for the trajectories never leaving Q_λ, X_n converges in probability to the support set of $\mathcal{M}_\infty$.*

Theorem A.4. *Let X_k be a Markov chain on the compact state space S'. Suppose that there exists a non-positive function $V(X)$ satisfying*

$$\mathbb{E}\left[V(X_{k+1})|X_k\right] - V(X_k) = -Q(X_k), \tag{A.1}$$

where $Q(X)$ is a positive continuous function of X, then the ω-limit set Ω (in the sense of almost sure convergence) of X_k is contained by the following set

$$I_\infty = \{X| \quad Q(X) = 0\}.$$

Proof. The proof is just an application of Theorem 1 in [55, Chapter 8], which shows that $Q(X_k)$ converges to zero for almost all paths. It is clear that the continuity of $Q(X)$ with respect to X and the compactness of S' implies that the ω-limit set of X_k is necessarily included into the set I_∞. □

A.3 Lyapunov exponents of linear stochastic processes

Consider a discrete-time linear stochastic system defined on $\mathbb{R}^d$ by

$$X_{k+1} = A_{\mu_k} X_k,$$

where A_{μ_k} is a random matrix taking its values inside a finite set $\{A_1, \ldots, A_m\}$ with a stationary probability distribution for μ_k over $\{1, \ldots, m\}$. Then

$$\lambda(X_0) = \lim_{k \to \infty} \frac{1}{k} \log\left(\frac{\|X_k\|}{\|X_0\|}\right),$$

for different initial states $X_0 \in \mathbb{R}^d$, may take at most d values which are called the Lyapunov exponents of the linear stochastic system.

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Titre : Stabilisation des systèmes quantiques à temps discret et stabilité des filtres quantiques à temps continu

Résumé : Dans cette thèse, nous étudions des rétroactions visant à stabiliser des systèmes quantiques en temps discret soumis à des mesures quantiques non-destructives (QND), ainsi que la construction et la stabilité de filtres quantiques à temps continu en présence d'imperfections. Cette thèse comporte deux parties.

Dans une première partie, nous généralisons les méthodes mathématiques sous-jacentes à une rétroaction quantique en temps discret testée expérimentalement au Laboratoire Kastler Brossel (LKB) de l'École Normale Supérieure (ENS) de Paris. Nous présentons d'abord l'algorithme de contrôle qui a été utilisé lors de cette expérience. Son but est la préparation et la stabilisation d'état à nombres entiers de photons (états de Fock) pour un champ micro-onde au sein d'une cavité supraconductrice. Pour cela, un filtre donne une estimation en temps-réel de l'état quantique malgré des imperfections et des retards de mesure. Cet état estimé est alors utilisé dans une loi de rétroaction (feedback) assurant la stabilisation vers un état de Fock prédéterminé. Cette stabilisation est obtenue grâce à des méthodes Lyapunov stochastiques. Nous montrons ici comment une telle stratégie de contrôle se généralise à d'autres systèmes quantiques en temps discret soumis à des mesures QND.

Dans une seconde partie, nous nous inspirons du temps discret pour construire et étudier la stabilité de filtres quantiques en temps continu et prenant en compte des imperfections de mesure. Tout d'abord, nous démontrons la stabilité d'un filtre quantique gouverné par l'équation maîtresse stochastique associée à un processus de Wiener. La stabilité signifie ici que la “distance" entre l'état physique et le filtre quantique décroit en moyenne. Cette distance est ici donnée par la fidélité, l'état physique et l'état du filtre pouvant être tous les deux des états mixtes. Cette partie étudie également la conception d'un filtre optimal en temps continu en présence d'imperfections de mesure. Pour ce faire, nous étendons la méthode utilisée précédemment pour construire les filtres quantiques en temps discret prenant en compte les imperfections de mesure. Nous obtenons alors heuristiquement des filtres généraux en temps continu, dont la dynamique est décrite par des équations maîtresses stochastiques mélangeant processus de Poisson et de Wiener. Nous conjecturons que ces filtres sont optimaux et stables.

Mots clés : Contrôle non-linéaire, mesures quantiques non-destructives (QND), électro-dynamique quantique en cavité (CQED), filtrage quantique, équation de Lindblad, équations maîtresses stochastiques (SMEs), équations de Schrödinger stochastiques (SSEs), méthode de Lyapunov, contrôle stochastique, rétroaction quantique , état de Fock.

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Title: Stabilization of discrete-time quantum systems and stability of continuous-time quantum filters

Abstract: In this thesis, we study measurement-based feedbacks stabilizing discrete-time quantum systems subject to quantum non-demolition (QND) measurements and stability of continuous-time quantum filters. This thesis contains two parts.

In the first part, we generalize the mathematical methods underlying a discrete-time quantum feedback experimentally tested in Laboratoire Kastler Brossel (LKB) at Ecole Normal Supérieure (ENS) de Paris. In fact, we contribute to a control algorithm which has been used in this recent quantum feedback experiment. This experiment prepares and stabilizes on demand photon-number states (Fock states) of a microwave field in a superconducting cavity. We design real-time filters allowing estimation of the state despite measurement imperfections and delays, and we propose a feedback law which ensures the stabilization of a predetermined target state. This stabilizing feedback is obtained by stochastic Lyapunov techniques and depends on a filter estimating the quantum state. We prove that such control strategy extends to other discrete-time quantum systems under QND measurements.

The second part considers an extension, to continuous-time, of a stability result for discrete-time quantum filters. Indeed, we prove the stability of a quantum filter associated to usual stochastic master equation driven by a Wiener process. This stability means that a “distance” between the physical state and its associated quantum filter decreases in average. Another subject that we study in this part is related to the design of a continuous-time optimal filter, in the presence of measurement imperfections. To this aim, we extend a construction method for discrete-time quantum filters with measurement imperfections. Finally, we obtain heuristically generalized continuous-time optimal filters whose dynamics are given by stochastic master equations driven by both Poisson and Wiener processes. We conjecture the stability of such optimal filters.

Keywords: Non-linear control, quantum non-demolition (QND) measurements, cavity quantum electrodynamics (CQED), quantum filtering, Lindblad equations, stochastic master equations (SMEs), stochastic Schrödinger equations (SSEs), Lyapunov-based techniques, stochastic control, measurement-based feedback, photon-number states (Fock states).

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